# Transport and Diffusion in Disordered Media

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*Academic dissertation*

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# In this thesis

This thesis consists of an introduction and the following two research papers:

(A) *Rigorous scaling law for the heat current in disordered harmonic chain*, with François Huveneers, Comm. Math. Phys. **301**, 841-883 (2011) [1]

(B) *Random walks in dynamic environments with integrable correlations* (preprint)

Some key ideas for paper (B) have been contributed by Wojciech De Roeck and Antti Kupiainen.

[1] 

# Introduction

## Contents

# 1 Prologue

It is a fundamental paradigm of physics that macroscopic phenomena, such as heat escaping through the wall of an apartment in winter, or the complicated motion of water masses flowing through a river, can be described at the level of molecules and atoms in terms of very large dimensional differential equations[2]. While this is theoretically intriguing, solving the equations of motions for such a huge set of non-linear equations is practically impossible. Fortunately therefore, it has been known for a long time that certain macroscopically relevant observables, typically densities corresponding to *conserved quantities* such as energy, momentum, number of particles, etc., satisfy autonomous and relatively simple partial differential equations (PDEs) to a very high accuracy. As an example, the heat is redistributed in ordinary materials by a heat equation, while the flow of fluids is determined by a somewhat more complicated set of Navier-Stokes equations. Both the heat and the Navier-Stokes equations are examples of *transport equations*. These are effective equations which describe how the macroscopic densities of (locally) conserved quantities are redistributed in space as a function of time.

For mathematical considerations the separation of the microscopic and the macroscopic scales is idealized as being infinitely large. Moreover, in order to be able to observe the desired macroscopic phenomena still after this *thermodynamic limit* is taken, space and time have to be scaled appropriately with respect to each others. The combined process of applying the appropriate scaling and the thermodynamic limit constitutes what is known as a *hydrodynamic limit*. Mathematically, the transport equations may be then valid only at the proper hydrodynamic limit.

Besides being autonomous, the transport equations are in general *irreversible*, i.e., they apparently loose information as time progresses, unlike the underlying reversible microscopic laws. In particular, transport equations may exhibit diffusion and, depending on the boundary conditions, relaxation towards an equilibrium or a steady state. Moreover, they exhibit *universality* in the sense that the form of the transport equation is determined only by a few *relevant parameters*, such as the dimension of the space variables, symmetries, etc., of the underlying microscopic models. Most of the microscopic details enter into these macroscopic descriptions only through a few phenomenological constants such as heat conductance or Reynolds-number. In nature the universality has striking consequences: the transport of heat in both standing water and crystalline solid are described by similar heat equations, while the flow of gases in an automotive engine and the lava erupting from a volcano are both described by Navier-Stokes equations. The implications of this universality hypothesis can also be used to justify research of toy models. Namely, it should be possible to understand the behavior of a realistic microscopic model at the hydrodynamic limit by studying overly simplified microscopic models, as long as they just belong to the right *universality class*.

A mathematically rigorous derivation of even the simplest transport equation from the first-principles, i.e., from the fundamental laws at the microscopic level, is still lacking. More precisely, so far nobody has succeeded in deriving a transport equation without introducing qualitatively unrealistic extra terms or making some uncontrolled approximations of the microscopic laws. Moreover, although there are rather convincing heuristic explanations of how, say, a finite heat conductivity results from microscopic collisions of heat carriers, even the question whether the change of spatial dimension

[2]Here we assume that classical physics is sufficient at the microscopic level.

changes the supposed hydrodynamic limit of a given model is still debated.

One particularly interesting microscopic feature that is generally known to have dramatic effects on the large scale properties of statistical mechanical systems, e.g., transport equations, is *disorder*. Disorder typically refers to general spatiotemporal irregularities, such as crystal defects, impurities, and space-time noise acting on the microscopic system which are fixed, or independent, of the rest of the variables. Disorder is mathematically modeled by making microscopic equations of motions random. As disordered systems are in general much more difficult to analyze than analogous ordered systems, it is not surprising that in the context of deriving transport equations the role played by disorder is in general poorly understood. As basically any real material is at least weakly disordered, in principle one can not even rule out the possibility that disorder might actually be a necessary ingredient for the derivation of some transport equations. Therefore, understanding the role played by disorder is of uttermost importance for modeling realistic models of physical as well as other complex systems.

The scientific contribution of this thesis consists of two papers (A) and (B) in which the effect of disorder on two mathematically well understood microscopic models of transport is investigated.

In paper (A), a disordered version of a harmonically coupled chain of oscillators is studied in one dimension. This is perhaps the simplest and one of the only solvable purely mechanical (Hamiltonian) models of heat conduction. The key result of paper (A) is the proof of the scaling law for the heat conductance of the chain as a function of the length of the chain. While in the ordered (non-disordered) case the heat conductivity of the chain is infinite, in the disordered case it becomes zero. Furthermore, it is shown that this behavior is universal in the sense that it is not affected by many details of the statistical properties or the magnitude of disorder. On the other hand, the conductivity is shown to depend strongly on how the heat is delivered to the chain at its boundaries - a property that is not expected to apply for anharmonic oscillators in general.

In paper (B), we consider a disordered version of the simplest microscopic model of heat conduction. Namely in this model, energy is carried by a large number of non-interacting random walkers on a multi-dimensional integer lattice, where the transition probabilities of the walkers depend on their position in space and time. Due to the lack of interactions between the heat carriers, it suffices to consider only a single walker in a dynamic random environment. We show that if the transition probabilities are correlated in an integrable manner in time, while the correlations decay weakly as function of increasing spatial separation, then the walker resembles a Brownian motion in the scaling limit for almost every realization of the disorder. In particular, this implies that a regular linear heat equation describes the density of the heat carriers at the large scale despite the presence of strongly correlated disorder in space. This result should be contrasted with what happens in one-dimensional static environments. In such case, any disorder which does not create an average drift will completely change the scaling behavior of the random walker. In particular, there is no diffusion in that case.

In summary, the results in papers (A) and (B) demonstrate two contrasting effects of the disorder on simple models of heat transport: In the first case, the addition of disorder changes the macroscopic behavior completely, while in the second case, it does not have a qualitative effect at all. Moreover, the technical analysis of paper (B) is based on renormalization group (RG) techniques introduced originally in particle and statistical physics contexts (cf. [17]). The RG paradigm offers an elegant way to understand the

origin of universality observed in the scaling limits of various microscopic models which extend infinitely in spacetime. In RG terminology, the result of paper (A) corresponds to proving the existence of a stable universal class of heat conduction of one-dimensional disordered mechanical chains. The results of paper (B) correspond to showing that the domain of attraction of Gaussian fixed points of random environments includes more general type of random environments than were previously known.

In the following pages we will provide heuristics as well as some supporting results for the papers. The intention is to communicate the key ideas without focus on details. We will also provide some general references although most of them can be found from papers (A) and (B) already.

# 2 Transport of heat by coupled mechanical oscillators

Consider a macroscopic system described by a temperature distribution $T_0(x)$, $x \in \bar{\Lambda} \subset \mathbb{R}^d$, at some given time $t = 0$. If the temperature distribution is not uniform, then an energy current $J : \bar{\Lambda} \to \mathbb{R}^d$ redistributing the heat energy is generated. Due to the local conservation of energy and basic thermodynamics, it appears reasonable to assume that

$$c_v(T(x,t))\,\frac{\partial T}{\partial t}(x,t) \;=\; -\nabla \cdot J(x,t)\,, \tag{1}$$

where $c_v$ is a specific heat per unit volume. Thermodynamics also tells us that if the temperature field $T(x,t) = T_0$ is uniform then the heat currents vanish. Hence it is natural to assume that for small temperature gradients, the current is given by *Fourier's law*

$$J(x,t) \;=\; -\kappa(T(x))\,\nabla T(x,t)\,, \tag{2}$$

where the heat conductance $\kappa : (0,\infty) \to (0,\infty)$ is a smooth function. For Fourier's law to hold one must also assume that no mass flow or other mechanisms of energy transport besides heat conduction are effective. Furthermore, the effects related to variations of density and pressure have been ignored. Combining Fourier's law with (1) yields an autonomous description of $T$ in terms of a non-linear parabolic PDE known as the *heat equation*:

$$c_v(T)\,\frac{\partial T}{\partial t} \;=\; \nabla \cdot \big[\kappa(T)\nabla T\big]\,. \tag{3}$$

There are two distinct situations where the heat equation is observed to hold experimentally with high accuracy for a wide range of solids as well as fluids:

*Isolated domain with uneven initial temperature field:* A block of material occupying the spatial domain $\bar{\Lambda} \subset \mathbb{R}^d$ with smooth boundaries is isolated from the environment in such a way that no heat can flow in or out of $\bar{\Lambda}$. This corresponds to the evolution of (3) started from the initial field $T(x,0) := T_0(x)$, with the Neumann boundary condition $n_{\partial\bar{\Lambda}}(x) \cdot \nabla T(x,y) = 0$, where $n_{\partial\bar{\Lambda}}(x)$, $x \in \partial\bar{\Lambda}$, is the normal vector of the smooth boundary $\partial\bar{\Lambda}$. In the large time limit $t \to \infty$, the temperature profile is expected to relax to the constant value, i.e., the system is expected to return to its equilibrium.

*Isolated domain subjected to non-uniform heat baths at the boundaries:* The second option is to consider a system subject to external heat baths. In that case the boundary

is split into two disjoint sets $\partial\bar{\Lambda} = \Gamma_0 \cup \Gamma_1$ where the part $\Gamma_0$ is isolating like in the previous case, while on the remaining part $\Gamma_1$ the temperature $T(x,t)$, $x \in \Gamma_1$, is fixed to a time-independent value $T_{\Gamma_1}(x)$ of the heat bath at $x$. For large times the system is expected to approach a stationary state where the temperature field is specified by

$$\nabla \cdot J \;\equiv\; \nabla \cdot \big[\kappa(T)\nabla T\big] \;=\; 0\,.$$

## 2.1 Coupled oscillators as microscopic model of energy transport

Perhaps the simplest realistic model of heat conduction at the microscopic scale consists of mechanical oscillators indexed by the elements of the set

$$\Lambda \;\equiv\; \Lambda_L \;:=\; \big\{i \equiv (i^1,\dots,i^d) \in \mathbb{Z}^d : i/L \in \bar{\Lambda}\big\}\,, \tag{4}$$

of lattice points, where $L$ describes the separation of microscopic and macroscopic length scales. Indeed, suppose the oscillator at site $i$ has mass $m_i \in (0,\infty)$, is located at $i + q_i \in \mathbb{R}^d$, and has momentum $p_i \in \mathbb{R}^d$. In the case of isolated boundary conditions, the evolution of the system is determined once the Hamiltonian

$$H(\mathrm{q},\mathrm{p}) \;\equiv\; H_L(\mathrm{q},\mathrm{p}) \;:=\; \sum_{i\in\Lambda_L} \frac{p_i^2}{2m_i} \;+ \sum_{i\in\Lambda_L} U_i(q_i) \;+ \sum_{\substack{i,j\in\Lambda_L \\ i\neq j}} V_{ij}(q_i - q_j)\,, \tag{5}$$

and the initial conditions of $\mathrm{q} = (q_i : i \in \Lambda), \mathrm{p} = (p_i : i \in \Lambda) \in (\mathbb{R}^d)^\Lambda$ are specified. Here the *onsite potential* $U_i : \mathbb{R}^d \to [0,\infty)$ pins the oscillator $i$ down to the crystal point $i$, while the *interaction potential* $V_{ij} : \mathbb{R}^d \to [0,\infty)$ models the coupling between oscillators $i$ and $j$. It is natural to assume that $U_i(q)$ and $V_{ij}(q)$ grow as $|q|$ increases, and that the interaction strength $V_{ij}(q)$ decreases as the atom separation $|i-j|$ grows. Based on the understanding of real matter as well as for the sake of convenience, it is common to assume that $V_{ij}$ vanishes unless $|i-j|_1 := \sum_{k=1}^d |i^k - j^k| = 1$.

Typically additional constraints on the potentials are needed to ensure that the mechanical system described by (5) are mathematically meaningful. For example, the equilibrium Gibbs probability measure with density $\exp(-T_0^{-1}H(\mathrm{p},\mathrm{q}))$ w.r.t. Lebesgue measure $\mathrm{dp}\,\mathrm{dq}$ at the constant temperature field $T(x) := T_0 > 0$ is well defined, provided that $U_i$ is non-zero for the oscillators at the boundary $\partial\Lambda$. In order to consider disordered systems later on, we have allowed the masses and the potentials to depend on the sites $i, j$. Although, even for ordered systems site depended potentials are often used to model boundary effects, in this subsection the site dependence can be ignored.

In order to connect this microscopic description to the heat equation we introduce *local energy at site* $i \in \Lambda$,

$$h_i(\mathrm{u}) \;:=\; \frac{p_i^2}{m_i} + U_i(q_i) + \frac{1}{2}\sum_j V_{ij}(q_i - q_j)\,, \tag{6}$$

with $\mathrm{u} := (\mathrm{q},\mathrm{p})$. The value of $\mathrm{u}(t) := (\mathrm{q}(t),\mathrm{p}(t))$ is determined uniquely by $\mathrm{u}(t) = \Phi_t(\mathrm{u}(0))$ where $\{\Phi_t\}_{t\in\mathbb{R}}$ is a diffeomorphic flow generated by the Hamiltonian equations, e.g., $\Phi_0 = \mathrm{Id}$, $\Phi_t^{-1} = \Phi_{-t}$ and $\Phi_s \circ \Phi_t = \Phi_{s+t}$. If the initial condition $\mathrm{u}(0)$ is chosen randomly it is natural to consider $t \mapsto \mathrm{u}(t)$ as a stochastic process since the flow $\Phi$ is typically chaotic.

The connection to the macroscopic temperature profile is now made through the empirical measure on spacetime corresponding to the local energies:

$$\tau_L(\mathrm{d}x\,\mathrm{d}t) \;:=\; L^{-d}\sum_{i\in\Lambda_L} h_i\big(\Phi_{L^2t}(\mathrm{u}(0))\big)\,\delta_{\{i/L\}}(\mathrm{d}x)\,\mathrm{d}t\,. \tag{7}$$

Let us also introduce the *local thermal equilibrium measures* on $(\mathbb{R}^d)^{\Lambda_L}\times(\mathbb{R}^d)^{\Lambda_L}$,

$$\nu_{T;L}(\mathrm{du}) \;:=\; Z_L(T)^{-1}\exp\Big(-\textstyle\sum_{i\in\Lambda_L} T(i/L)^{-1}h_i(\mathrm{u})\Big)\,\mathrm{du}\,, \tag{8}$$

where $T:\bar\Lambda\to[0,\infty)$ is a smooth function. Suppose $\mathrm{u}(0)$ is distributed according to $\nu_{T_0;L}$ for a smooth initial temperature field $T_0:\bar\Lambda\to(0,\infty)$, and assume for simplicity that $U_i:=U\neq 0$ and $V_{ij}:=V$ do not depend on $i,j$. A full proof of Fourier's law would consist of showing for any smooth test function $f:\bar\Lambda\times[0,1]\to\mathbb{R}$, that the limit

$$\lim_{L\to\infty}\iint f(t,x)\,\tau_L(\mathrm{d}x\,\mathrm{d}t) \;=\; \iint f(x,t)T(x,t)\mathrm{d}x\,\mathrm{d}t\,, \tag{9}$$

applies for $\nu_{T_0}$-a.e. initial value $\mathrm{u}(0)$, with the limiting temperature field $T:\bar\Lambda\times[0,1]\to[0,\infty)$ satisfying a non-linear heat equation (3) with initial density $T(x,0)=T_0(x)$.

The physical meaning of this statement is that the microscopic degrees of freedom around $(x,t)$ are well described by the Gibbs measure corresponding to the local temperature $T(x,t)$ at the thermodynamic limit. This is intuitively plausible since the smooth macroscopic temperature $T(x,t)$ varies at the microscopic scales of space and time with derivatives of order $L^{-1}$ and $L^{-2}$, respectively. Nevertheless, the rigorous justification of this *local thermal equilibrium* (LTE) hypothesis has turned out to be an extremely challenging problem (cf. [4]) for any (qualitatively) realistic microscopic model, such as the one corresponding to (5), of energy transport. Roughly speaking, the crux of the matter is to be able to justify various probabilistic estimates which would guarantee sufficient mixing and independence between the microscopic degrees of freedom as a function of their separations in spacetime. Such estimates are in turn needed when one replaces microscopic degrees of freedom around $(x,t)\in\bar\Lambda\times[0,\infty)$ by their local averages described by the local temperature $T(x,t)$.

Before moving on it is worth noting that one could consider a more general model of heat conduction where the particles are not fixed on the lattice, but can move freely in the container $L\bar\Lambda\subset\mathbb{R}^d$. Such a system corresponds to a model of liquid and could be described by the Hamiltonian

$$\widetilde H_L(\mathrm{q},\mathrm{p}) := \sum_{\alpha=1}^{\rho N^2}\left[\frac{p_\alpha^2}{2m}+U_L(q_\alpha)\right]+\sum_{\alpha,\beta=1:\alpha\neq\beta}^{\rho L^d} V(q_\alpha-q_\beta)\,,$$

where particle $\alpha$ has position and momentum $(q_\alpha,p_\alpha)\in(L\bar\Lambda)\times\mathbb{R}^d$, potential $U_L$ describes the interaction of the particles with the container of the liquid, and $\rho\sim 1$ is the average density of the particles. Here the potentials are assumed to be independent of $\alpha,\beta$ since we are not considering the disordered case. [3]

In the case of liquid the empirical spacetime measure of energy resembles (7), except that the Dirac-measures depend also on the positions of the particles, i.e.,

$$\tilde\tau_L(\mathrm{d}x\,\mathrm{d}t) \;:=\; L^{-d}\sum_{\alpha=1}^{\rho L^d}\tilde h_\alpha(\mathrm{q}(L^2t),\mathrm{p}(L^2t))\,\delta_{\{L^{-1}q_\alpha(L^2t)\}}(\mathrm{d}x)\,\mathrm{d}t\,. \tag{10}$$

[3] If potentials are allowed to depend on $\alpha,\beta$ it is possible to write the crystal Hamiltonian (5) in this form as well.

with

$$\tilde{h}_\alpha(\mathrm{q},\mathrm{p}) \;:=\; \frac{p_\alpha^2}{2m} + U_L(q_\alpha) + \tfrac{1}{2}\sum_\beta V(q_\alpha - q_\beta)\,.$$

We again expect that $\tilde{\tau}_L$ converges weakly to a field $T$ analogously to (9). However, the evolution of $T$ is not anymore described by heat equation (3). Instead, in order to obtain an autonomous hydrodynamic description for the system, one has to introduce similar hydrodynamic fields for each locally conserved quantity, e.g., linear and angular momentum, and to show that these fields together satisfy an autonomous set of partial differential equations commonly known as Navier-Stokes equations (cf. [28]).

## 2.2 Heat flow through driven system

If one considers a Hamiltonian system (5) interacting with heat baths, the energy current through the system can be studied without having to, at least directly, face the problems associated to proving the weak limit (9) or the LTE-hypothesis. In such system the time evolution of $\mathrm{u} \equiv (\mathrm{q},\mathrm{p})$ is governed by the Hamiltonian equations and the interactions with the heat baths at the boundary of $\Lambda_L$. Since it is expected, although not proven (cf. [15]), that the exact choice of the heat baths does not make a difference in physically relevant models, it is customary to model these interactions by stationary stochastic processes, such as Orstein-Uhlenbeck processes (Langevin-type noise). Such choice then yields the stochastic differential equations

$$\begin{aligned} \mathrm{d}q_i(t) \;&=\; \frac{\partial H_L}{\partial p_i}(\mathrm{u}(t))\,\mathrm{d}t \\ \mathrm{d}p_i(t) \;&=\; -\frac{\partial H_L}{\partial q_i}(\mathrm{u}(t))\,\mathrm{d}t \;+\; \mathbf{1}\{i \in \partial\Lambda_L\}\cdot\big(-\gamma_i p_i(t)\mathrm{d}t + \sqrt{2\gamma_i T_i m_i}\,\mathrm{d}B_i(t)\big)\,, \end{aligned} \tag{11}$$

where $\gamma_i \geq 0$ describes the strength of the coupling between the oscillator $i$ and the associated heat bath, $t \mapsto B_i(t) : [0,\infty] \to \mathbb{R}^d$ are independent copies of a standard $d$-dimensional Brownian motion, and $\partial\Lambda_L := \big\{i \in \Lambda_L : \exists j \in \mathbb{Z}^d\backslash\Lambda_L \text{ s.t. } |i-j|_1 = 1\big\}$.

One could attempt to prove the limit (9) and show that the field $T(x,t)$ satisfies the heat equation with boundary values $T_i \equiv T_{L;i} := T|_{\partial\bar{\Lambda}}(x)$, where $x$ is the closest point of $\partial\bar{\Lambda}$ from $i/L$. On the other hand, by differentiating the energy process $t \mapsto H(\mathrm{u}(t))$ and arranging the terms, we obtain

$$\begin{aligned} \mathrm{d}(H\circ\mathrm{u})(t) \;&=\; \sum_{i\in\Lambda}\left\{\frac{\partial H}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\mathrm{d}t\right) \;+\; \frac{\partial H}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\mathrm{d}t\right)\right\} \\ &\quad+\; \sum_{j\in\partial\Lambda}\left\{\frac{\partial H}{\partial p_j}\left(\mathrm{d}p_j + \frac{\partial H}{\partial q_j}\mathrm{d}t\right) \;+\; \frac{1}{2}\frac{\partial^2 H}{\partial p_j^2}\mathrm{d}\langle p_j\rangle\right\} \\ &=\; 0 \;+\; \sum_{j\in\partial\Lambda}\left\{\gamma_j\left(T_j - \frac{p_j^2(t)}{m_j}\right)\mathrm{d}t \;+\; \frac{2\gamma_j T_j}{m_j}p_j(t)\,\mathrm{d}B_j(t)\right\}\,, \end{aligned}$$

where each term on the last line corresponds to a boundary site $j$ and is identified as the instantaneous energy current from the heat baths to the system of oscillators.

If one assumes that there is a unique ergodic stationary measure $\nu_L$ for the process (11), then taking expectations w.r.t. this measure yields the formula for the stationary energy current through a boundary site,

$$\bar{J}_i \;:=\; \gamma_i\left(T_i \;-\; \frac{1}{m_i}\int p_i^2\,\nu_L(\mathrm{du})\right)\,, \qquad i \in \partial\Lambda\,. \tag{12}$$

Here, $\bar{J}_i > 0$ when energy is in average flowing from the heat baths to the system of oscillators at the site $k$. By assumption of stationarity, one obtains $\sum_{i\in\partial\Lambda} \bar{J}_i = 0$.

A simple formula for the heat conductance of the total system is obtained by choosing

$$\bar{\Lambda} \,=\, [0,1]^d\,, \qquad \text{so that} \qquad \Lambda_L \,=\, \{1,2,\ldots,L\}^d\,,$$

and setting the system in contact with the heat baths of temperatures $T$ and $T'$ at the $(d-1)$-dimensional surfaces corresponding the extreme values of the first coordinate $i^1 = 1$ and $i^1 = L$, respectively, while isolating the remaining boundary sites $i$ by setting $\gamma_i = 0$.

Since in the stationary state the expected total energy does not change, we obtain from (12) an expression for the total current carried flowing through the system in the positive direction along the first coordinate axis,

$$\mathcal{J}_L^{11}(T,T') \;:=\; \sum_{i\in\partial\Lambda_L : i^1=1} \bar{J}_i\,, \tag{13}$$

where $\mathcal{J}_L^{ij}(T,T')$, $1 \le i,j \le d$, denotes the current in direction $i$ when there is a heat gradient in direction $j$. By normalizing the total current by the cross section of the conducting block, i.e., by the area $L^{d-1}$ of the surface $\{i \in \partial\Lambda : i^1 = 1\}$, as well as dividing by the temperature gradient $L^{-1}(T-T')$, we obtain for the average heat conductivity between temperatures $T > T'$ the expression

$$\bar{\kappa}_L^{11}(T,T') \;=\; \frac{L^{2-d}}{T-T'}\,\mathcal{J}_L^{11}(T,T')\,. \tag{14}$$

Taking the thermodynamic limit $L \to \infty$ yields a formula for the corresponding thermodynamic average conductivity. By letting also $T'$ approach $T$ yields a microscopic definition for the conductivity appearing in the non-linear heat equation (3). More precisely,

$$\bar{\kappa}(T,T') \;:=\; \lim_{L\to\infty} \bar{\kappa}_L^{11}(T,T')\,, \qquad \kappa(T) \;:=\; \lim_{T'\nearrow T} \bar{\kappa}(T,T')\,, \tag{15}$$

where it has been assumed that the system is symmetric w.r.t. rotations and translations at the macroscopic scale, and hence tensors have been reduced to scalars.

If one were able to show that $C^{-1} \le L^{2-d}\mathcal{J}_L^{11}(T,T') \le C$, uniformly as $L \to \infty$, for some $0 < T, T' < \infty$, that would be considered already as a proof of Fourier's law. After all, if one supposes $\kappa(T'') \sim \kappa(T)$, for every $T \le T'' \le T'$, then this scaling law would imply the finite conductivity $\kappa(T) \sim 1$ in (2). Moreover, as already mentioned, by using formulas (14) and (15) one sidesteps the problem of justifying the LTE hypothesis, and reduces the problem to that of calculating the second moments $p_i$'s under the stationary measure [4].

In one dimension the oscillators form a chain with $\Lambda := \{1,2,\ldots,L\}$. In this case, $\partial\Lambda = \{1,L\}$, and hence it follows $\bar{J}_1 = -\bar{J}_L$. Moreover, (12) implies that the total energy current carried by the chain from left to right is

$$\begin{aligned} \mathcal{J}_L(T_1,T_L) \;&:=\; \lambda_1 T_1 \,-\, (\lambda_1/m_1)\,\nu_{L;T_1,T_L}\big(p_1^2\big) \\ &=\; (\lambda_L/m_L)\,\nu_{L;T_1,T_L}\big(p_L^2\big) \,-\, \lambda_L T_L\,, \end{aligned} \tag{16}$$

[4] There is also a third, more practical way to define $\kappa(T)$ using Green-Kubo formula (see [28]).

where we have reindexed the stationary $\nu_L$ so that its temperature dependence is explicit $\nu_{L;T_1,T_L}$.

Although the verification of the local thermal equilibrium remains still open, the existence of unique stationary mixing measure $\nu_L$ for (11) has been proven with reasonably general potentials for equal masses $m_i = m$ in the one-dimensional setting in [13]. Moreover, in the same work it is shown that the current vanishes under these conditions if and only if $T_1 = T_L$. The difficulties of dealing with the system (11) in general are related to showing that the system is mixing enough to justify various probabilistic assumptions. Although there are purely stochastic terms present at the boundaries, they are very far away from an average oscillator as $L$ approaches infinity.

It is worth remarking that it is now generally believed that in the absence of an external pinning potential, as is the case in most realistic situations, the heat conductance in one dimension is not finite. This is an important finding also from practical perspective, as it describes the heat transport of carbon nanotubes and has been recently even studied experimentally (cf. [10]).

## 2.3 Disorder

Because of the apparent translational invariance of materials it would be natural to assume that $m_i$ and $U_i$ do not depend on $i$, while the interparticle potential $V_{ij}$ would depend only on the relative equilibrium position $i-j$ of the particles in the Hamiltonian (5). However, we want to consider disordered models where impurities or non-perfect crystal structures invalidate such symmetry properties. On the other hand, we are not interested in how the system behaves for a specific disordered Hamiltonian, but want to instead understand how a *typical* disordered Hamiltonian behaves when it is thought to originate from a statistical ensemble. This is modeled by assuming the Hamiltonian to be a random variable $\omega \mapsto H^\omega$ [5] on some probability space $(\Omega, \Sigma, \mathbb{P})$.

The simplifying symmetries of the ordered Hamiltonian are then transferred to corresponding assumptions on the probability distribution of the Hamiltonian, e.g., $m_i$ are i.i.d. random variables, or more generally, the local energies (6) $(h_{i+j} : i \in I)$ at sites $I \subset \mathbb{Z}^d$ have the same law as $(h_i : i \in I)$ in the limit $\Lambda_L \to \mathbb{Z}^d$. Since typical macroscopic observables are averages over a very large number of the microscopic degrees of freedom, their randomness is often expected to be averaged out, or *homogenized*, at the scaling limit. In particular, the conductivities are expected to become non-random and inherit the symmetries satisfied by the statistics for typical realizations of the disorder at the limit $L \to \infty$. This has been anticipated in (15) by writing $\kappa(T)$ and $\bar\kappa(T, T')$ as scalars instead of tensors.

Since the Hamiltonian is random also the associated steady state currents and conductivities are random variables. As is customary with the disordered system one talks about *annealed* and *quenched* properties of the system. As an example of this terminology, one says that the average heat conductivity of (14) corresponding to the disordered Hamiltonian $H_L$ scales like $L^\alpha$ in a quenched sense, provided

$$C^{-1} \;\le\; \liminf_{L\to\infty} L^{-\alpha}\, \bar\kappa_L^{11;\omega}(T,T') \le \limsup_{L\to\infty} L^{-\alpha}\, \bar\kappa_L^{11;\omega}(T,T') \;\le\; C\,, \qquad \mathbb{P}-\text{a.e. } \omega\,,$$

for a constant $C > 0$. On the other hand, one says that $\bar\kappa_L^{11}(T,T') \sim L^\alpha$ in an annealed

[5] This is equivalent to saying that all or some of the quantities $m_i, U_i, V_{ij}$ are random.

sense if

$$C^{-1} \;\le\; \liminf_{L\to\infty} L^{-\alpha}\,\mathbb{E}\big[\bar\kappa_L^{11}(T,T')\big] \;\le\; \limsup_{L\to\infty} L^{-\alpha}\,\mathbb{E}\big[\bar\kappa_L^{11}(T,T')\big] \;\le\; C\,.$$

In general the effect of disorder on $\bar\kappa_L^{11}(T,T')$ is not well understood. Because of the complicated finite size effects, simulations have not provided a clear picture either. For details concerning the current state of knowledge see review paper [10].

## 2.4 Disordered harmonic systems and localization of eigenmodes

If the potentials are quadratic in their arguments, i.e., there exist constants $U_i^{kl}, V_{ij}^{kl} \ge 0$, such that

$$U_i(q) \;=\; \sum_{k,l=1}^{d} U_i^{kl} q^k q^l\,, \qquad V_{ij}(r) \;:=\; \sum_{k,l=1}^{d} V_{ij}^{kl} r^k r^l\,, \qquad i,j\in\Lambda_L\,,$$

we say that Hamiltonian (5) is *harmonic*, and otherwise *anharmonic*. Let us assume for convenience, and without missing anything important, that the deviations of the oscillators in the Hamiltonian (5) are one-dimensional, i.e., $q_i, p_i \in \mathbb{R}$. In this case we may also drop the superscripts from the potentials, so that $U_i := U_i^{11}$ and $V_{ij} := V_{ij}^{11}$. Furthermore without loosing any generality one may assume that $V_{ji} = V_{ij}$.

The pure Hamiltonian equations now reduce to the linear Newton equations

$$M\ddot{\mathrm{q}}(t) \;=\; -\Phi \mathrm{q}(t)\,, \tag{17}$$

where the mass matrix $M$ and the force matrix $\Phi$ are symmetric matrices in $\mathbb{C}^{\Lambda_L\times\Lambda_L}$ defined by

$$M_{ij} := m_i\delta_{ij}\,, \qquad \Phi_{ij} \;:=\; (2U_i + 4V_{ii})\delta_{ij} \;-\; 4V_{ij}\,, \qquad i,j\in\Lambda_L\,.$$

Since $M^{-1}\Phi$ is positive w.r.t. the inner product [6]

$$\langle \mathrm{q}, \mathrm{q}'\rangle_M \;:=\; \mathrm{q}^\dagger M \mathrm{q}'\,, \qquad \mathrm{q},\mathrm{q}'\in\mathbb{C}^{\Lambda_L}\,,$$

there exist eigenvalues $\phi_\alpha^2 > 0$, and the corresponding eigenvectors $\mathrm{q}_\alpha \in \mathbb{C}^{\Lambda_L}$, $\alpha = 1,\dots,N := L^d$, such that

$$(M^{-1}\Phi)\mathrm{q}_\alpha \;=\; \phi_\alpha^2 \mathrm{q}_\alpha\,, \qquad \langle \mathrm{q}_\alpha, \mathrm{q}_\beta\rangle_M \;=\; \delta_{\alpha\beta}\,, \qquad 1\le\alpha,\beta\le N\,. \tag{18}$$

Suppose, the Newton equations (17) are started from $\mathrm{q}(0) := \mathrm{q}'$ and $\mathrm{p}(0) := \mathrm{p}'$, and denote the projections to the normal modes by $\Pi_\alpha := \mathrm{q}_\alpha \mathrm{q}_\alpha^\dagger$. An explicit solution can now be written in terms of $\mathbb{C}^{\Lambda_L\times\Lambda_L}$-block matrices,

$$\begin{bmatrix} \mathrm{q}(t) \\ M^{-1}\mathrm{p}(t) \end{bmatrix} \;=\; \sum_{\alpha,\sigma} \begin{bmatrix} \cos\phi_\alpha t\;\Pi_\alpha & \sigma\phi_\alpha^{-1}\sin\phi_\alpha t\;\Pi_\alpha \\ -\sigma\phi_\alpha\sin\phi_\alpha t\;\Pi_\alpha & \cos\phi_\alpha t\;\Pi_\alpha \end{bmatrix} \begin{bmatrix} M\mathrm{q}' \\ \mathrm{p}' \end{bmatrix}\,, \tag{19}$$

with $\alpha$ and $\sigma$ summed over all $N \equiv L^d$ eigenmodes and signs $\{+1,-1\}$, respectively.

Suppose now that we add a driving term $\mathrm{f}:\mathbb{R}\to\mathbb{R}^{\Lambda_L}$ into Newton's equation:

$$M\ddot{\mathrm{q}}(t) \;=\; -\Phi\mathrm{q}(t) \;+\; \mathrm{f}(t)\,, \qquad \mathrm{q}(0) = \mathrm{p}(0) = 0\,.$$

[6] Here $\mathrm{q}^\dagger$ is obtained from q by transposing and taking complex conjugates componentwise.

From (19) one obtains by applying Duhamel's formula the exact solution

$$\mathrm{p}(t) \;=\; 2\sum_{\alpha}\int_0^t \cos\phi_\alpha(t-s)\; M\Pi_\alpha \mathrm{f}(s)\,\mathrm{d}s\,. \tag{20}$$

In order to understand the heat transport of harmonic systems, let us pick two arbitrary sites $i, j \in \Lambda$, and drive the oscillator at site $i$ by an $L^2$-function $f : \mathbb{R} \to \mathbb{R}$ by setting $\mathrm{f}(t) := f(t)\mathrm{e}_i$. If we assume that $f$ has also a bounded support, then a simple calculation on frequency domain reveals how this injection of energy at site $i$ affects the oscillator at site $j$

$$\mathcal{E}_{L;ij}(f) \;:=\; \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau p_j(t)^2\mathrm{d}t \;=\; m_j^2\sum_\alpha (q_{\alpha;j})^2(q_{\alpha;i})^2|\hat{f}(\phi_\alpha)|^2\,, \tag{21}$$

where $\hat{f}(\phi) := \int \mathrm{e}^{-\mathrm{i}\phi t}f(t)\mathrm{d}t$ and $\mathrm{q}_\alpha = (q_{\alpha;i} : i \in \Lambda_L)$ so that $\mathrm{e}_i^{\mathrm{T}}\mathrm{q}_\alpha = q_{\alpha;i}$. The expression on the right hand side now shows that the effect felt at $j$ crucially depends on how spread out the eigenvectors are on $\Lambda_L$. Indeed, suppose for simplicity that $f$ is chosen such that $\hat{f}(\phi_\alpha) \sim 1$, i.e., $f$ is qualitatively similar to the white noise on the spectrum $\{\phi_\alpha : \alpha = 1, \ldots, N\}$. By assuming that (21), with $|i-j| \sim L$, scales like (14) when $L \to \infty$, the influence of disorder and spatial dimension on the conductivity is determined by the joint distributions of the pairs $(\phi_\alpha, \mathrm{q}_\alpha)$:

- For ordered systems it can be typically shown that all eigenvectors are completely *delocalized*, or *extended*, so that typically $(q_{\alpha;i})^2 \sim L^{-d}$ regardless of $i$. This implies that energy injected at $i$ is spread across the whole system, so that $\mathcal{E}_{L;ij}(f) \sim 1$, for every $j$, and the heat conductance is $\mathcal{O}(L)$.

- For disordered systems in one dimension, it can be shown that the eigenmodes are exponentially localized. By this, one means that as $L \to \infty$, the components $q_{\alpha;i}$ of the eigenvectors typically peak around some site $i_\alpha$, and then decay exponentially when one moves away from $i_\alpha$: $|q_{\alpha;j}/q_{\alpha;i_\alpha}| \lesssim \exp(-|j-i_\alpha|/\ell_\alpha)$. The *localization length* $\ell_\alpha$ is typically of order $\ell_\alpha \sim 1$. However, in the absence of pinning potential, for low frequencies one has $\ell_\alpha \sim \phi_\alpha^{-2}$. Hence, by (21) the modes $\phi_\alpha \lesssim L^{-1/2}$ carry heat effectively across the system (cf. next subsection).

- For disordered systems in two dimensions, rigorous results concerning the localization properties of the eigenvectors are lacking. Based on simulations and heuristics it is expected, but not proven, that only modes $\phi_\alpha \lesssim (\ln L)^{-1/2}$ carry effectively heat due to the associated localization lengths exceeding the linear size $L$ of the system;

- For disordered systems in three and higher dimensions, no rigorous theory exists. However, again by simulations and non-rigorous arguments, there are expected to be typically both extended as well as localized eigenmodes present with their relative proportions depending on strength and type of disorder. The extended vectors form a conducting band, and the width of this band determines the conductance of the whole system. The dependence of conductance on the statistical properties of the disorder can be complicated.

More information and references concerning the heat transport in disordered harmonic systems can be found from [9] and [10]. In physics, the localization phenomena of

the eigenvectors of large random band matrices, like $M^{-1}\Phi$, is known as *Anderson localization* (cf. [1]), and it has been studied vigorously in the context of non-interacting quantum mechanical electrons moving in a disordered potential.

Although harmonic models are known to have unphysical heat conductivity, they still serve as the starting point when one attempts to explain heat transport in anharmonic systems. For example, the prevailing physical theory by Peirls (cf. [4] for review and further references) explains the finite conductivity in terms of phonons, which are loosely speaking localized wavepackets corresponding to the eigenmodes of the harmonic approximation of the potentials appearing in the Hamiltonian. In Peirl's theory one explains (heuristically) the diffusion of energy as a result of collisions between the wavelike phonons caused by the anharmonic parts of the potentials.

Another example of the important role that the harmonic systems play in the theoretical understanding of the heat conductance is demonstrated in the works [2] and [3]. In the former paper the dynamics of an ordered harmonic system are augmented by diffusive noise which conserves both energy and momentum locally. In the latter paper, the dynamics of a disordered harmonic system are augmented by the diffusive noise which conserves energy locally. The physical motivation behind both of these models is that the added noise resembles, at the qualitative level, the chaotic dynamics associated with anharmonic potentials. The mixing generated by the noise on the other hand results in stochastic systems which can still be analyzed rigorously. Both papers yield results which correspond to the numerically observed behavior for the corresponding anharmonic systems. In particular, the conservative noise makes the conductivity of a disordered chain finite. Heuristically, this can be understood in terms of exchange of energy between the normal modes induced by the noise.

## 2.5 Derivation of heat current in disordered harmonic chains

For disordered harmonic systems only the one-dimensional case is currently rather well understood. In particular, a useful expression for the stationary heat current has been derived in [8]. Since the analysis of paper (A) builds on this result, we now reproduce it (Proposition 2.1 below) in a slightly different way using stochastic calculus, and the general formula (16) valid for chains.

For simplicity, we assume that only the masses $m_i$ are random, while setting $U_i(q) := \mu q^2$ and $V_{ij}(q) := \mathbf{1}\{|i-j|=1\}q^2$, so that (5) simplifies to

$$H(\mathrm{q},\mathrm{p}) \;=\; \sum_{k=1}^{L}\left(\frac{p_k^2}{2m_k}+\frac{\mu}{2}q_k^2\right) \;+\; \frac{1}{2}\sum_{k=0}^{L}(q_{k+1}-q_k)^2\,,$$

where we have either the free or fixed boundary conditions:

- Free boundaries: $q_0(t)=q_1(t)$ and $q_{L+1}(t)=q_L(t)$ for all times $t$;
- Fixed boundaries: $q_0(t)=q_{L+1}(t)=0$ or all times $t$.

The SDE (11) becomes in this setting an Ornstein-Uhlenbeck process

$$\mathrm{du}(t) \;=\; -K\mathrm{u}(t)\mathrm{d}t \;+\; D^{1/2}\mathrm{dB}(t)\,, \qquad \mathrm{u}(t)^{\mathrm{T}} = \left[\,\mathrm{q}(t)^{\mathrm{T}}\,\mathrm{p}(t)^{\mathrm{T}}\right], \tag{22}$$

where $\mathrm{B}(t) := (B_i(t) : 1 \le i \le 2n)$ is an $L$-dimensional standard Brownian motion, while the constant matrices $K, D \in \mathbb{R}^{2L\times 2L}$ are defined by

$$K = \begin{bmatrix} 0 & -M^{-1} \\ \Phi & \Gamma \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 \\ 0 & 2\Gamma TM \end{bmatrix}. \tag{23}$$

Here, the submatrices $M, \Gamma, T \in \mathbb{R}^{L\times L}$ are diagonal and positive

$$M := \mathrm{diag}(m_1, \ldots, m_n)\,, \qquad \Gamma := \mathrm{diag}(\gamma, 0, \ldots, 0, \gamma)\,, \qquad T := \mathrm{diag}(T_1, 0, \ldots, 0, T_n)\,,$$

i.e., $\gamma_1 = \gamma_L = 1$, while the force matrix $\Phi$ is triagonal (with $q_0 = q_{L+1} = 0$),

$$\Phi_{ij} \;\equiv\; \begin{cases} \Phi_{ij}^{(\mathrm{Free})} := (2 \,+\, \mu - \delta_{i1} - \delta_{iL})\delta_{ij} \,-\, \delta_{i,j+1} - \delta_{i,j-1} & \text{with free boundaries;} \\ \Phi_{ij}^{(\mathrm{Fix})} := (2 \,+\, \mu)\delta_{ij} \,-\, \delta_{i,j+1} - \delta_{i,j-1} & \text{with fixed boundaries.} \end{cases} \tag{24}$$

By using Duhamel's formula one may solve (22) so obtain:

$$\mathrm{u}(t) \;:=\; \mathrm{e}^{-tK}\,\mathrm{u}(0) \,+\, \textstyle\int_0^t \mathrm{e}^{-(t-s)K} D^{1/2} \mathrm{dB}(s)\,. \tag{25}$$

Since the integrand of the stochastic integral is nonrandom, the process is Gaussian. If one sets $\mathrm{B} = 0$ in (25), then one is left with a damped Hamiltonian system which, due to $-\gamma p_i$ terms, is dissipative, i.e., it looses energy as time progresses. In the context of (25) this means that the eigenvalues of $K$ have positive real components (cf. Lemma 2.2 below), and hence one obtains exponential damping in time

$$\|\mathrm{e}^{-tK}\| \;\lesssim\; \mathrm{e}^{-ct}\,, \quad t \geq 0\,, \tag{26}$$

with some $c > 0$. This in turn implies that only the stochastic integral in (25) has an effect on

$$\begin{aligned} \nu(\mathrm{u}\mathrm{u}^{\mathrm{T}}) \;&=\; \lim_{t\to\infty} E_{\mathrm{B}}\,\mathrm{u}(t)\mathrm{u}(t)^{\mathrm{T}} \\ &=\; 0 \,+\, \textstyle\int_0^\infty \mathrm{e}^{-tK} D \mathrm{e}^{-tK^{\mathrm{T}}} \mathrm{d}t\,, \end{aligned} \tag{27}$$

where $\nu \equiv \nu_{L;T_1,T_L}$ is the stationary measure of the process u, while $E_{\mathrm{B}}$ denotes the expectation w.r.t. driving Brownian motion B. In order to obtain (27) we have used the Itô-isometry and the independence of Brownian motions $B_1$ and $B_L$.

For convenience, we write

$$S(t) := \mathrm{e}^{-Kt} =: \begin{bmatrix} * & F(t) \\ * & G(t) \end{bmatrix} \tag{28}$$

where the stars $*$ denote unspecified $L \times L$-matrices. Although the matrices $\Phi, \Gamma, M, T$ are almost diagonal, the same is not true for $S(t)$, and we do not have a practical expression for the correlations in (27). However, by expressing the problem in the frequency domain we will obtain almost diagonal matrices.

To this end, we extend $S(t)$ for all $t \in \mathbb{R}$, so that

$$\dot{S}(t) \;=\; -KS(t)\,, \quad S(0) \;=\; I\,, \quad \text{and} \quad S(-t) \;:=\; 0\,, \quad t > 0\,.$$

By (26) we know that $S, \dot{S} \in L^p(\mathbb{R}; \mathbb{C}^{2L\times 2L})$, $1 \leq p \leq \infty$. Thus the Fourier transforms $\hat{S}, \widehat{\dot{S}} : \mathbb{R} \to \mathbb{C}^{2L\times 2L}$ are well defined, and by partial integration, we obtain

$$\widehat{\dot{S}}(\phi) \;=\; \textstyle\int_0^\infty \mathrm{e}^{-\mathrm{i}\phi t} \dot{S}(t) \mathrm{d}t \;=\; -I \,+\, \mathrm{i}\phi \int_0^\infty \mathrm{e}^{-\mathrm{i}\phi t} S(t) \mathrm{d}t \;=\; -I \,+\, \mathrm{i}\phi \hat{S}(\phi)\,. \tag{29}$$

Substituting this into $\dot{S}(t) = -KS(t)$ yields

$$(\mathrm{i}\phi I + K)\hat{S}(\phi) \;=\; I\,.$$

Writing this in terms of the $L \times L$-blocks $F(t), G(t)$ yields

$$\begin{aligned} \mathrm{i}\phi\hat{F}(\phi) \,-\, M^{-1}\hat{G}(\phi) \,&=\, 0\,, \\ \Phi\hat{F}(\phi) \,+\, (\mathrm{i}\phi I + \Gamma)\hat{G}(\phi) \,&=\, I\,. \end{aligned} \tag{30}$$

We may solve both $\hat{F}(\phi)$ and $\hat{G}(\phi)$ in terms of the symmetric triagonal matrix,

$$Z(\phi) \;:=\; \Phi - \phi^2 M + \mathrm{i}\phi\Gamma M\,, \tag{31}$$

which is invertible by Lemma 2.2 for every $\phi \in \mathbb{R}$:

$$\hat{F}(\phi) := Z(\phi)^{-1} \qquad \text{and} \qquad \hat{G}(\phi) := \mathrm{i}\phi M Z(\phi)^{-1}\,. \tag{32}$$

Let us express the correlation integral (27) in terms of the Fourier variables as well. Indeed, by Plancherel's formula we obtain

$$\nu(u_i u_j) \,\equiv\, \nu_{L;T_1,T_L}(u_i u_j) \;=\; \sum_{k=1}^{2L} D_{kk} \int_{\mathbb{R}} S_{ik}(t) S_{jk}(t)\mathrm{d}t \;=\; \sum_{k=1}^{2L} \frac{D_{kk}}{2\pi} \int_{\mathbb{R}} \hat{S}_{ik}(\phi)\hat{S}_{jk}(\phi)^* \mathrm{d}\phi\,.$$

Writing this in terms of $L \times L$-submatrices yields

$$\begin{bmatrix} \nu(\mathrm{q}\mathrm{q}^{\mathrm{T}}) & \nu(\mathrm{q}\mathrm{p}^{\mathrm{T}}) \\ \nu(\mathrm{p}\mathrm{q}^{\mathrm{T}}) & \nu(\mathrm{p}\mathrm{p}^{\mathrm{T}}) \end{bmatrix} \;=\; \frac{1}{2\pi} \int_{-\infty}^{\infty} \begin{bmatrix} \hat{F}(\phi)^\dagger D' \hat{F}(\phi)^\dagger & \hat{F}(\phi) D' \hat{G}(\phi)^\dagger \\ \hat{G}(\phi) D' \hat{F}(\phi)^\dagger & \hat{G}(\phi) D' \hat{G}(\phi)^\dagger \end{bmatrix} \mathrm{d}\phi\,,$$

where $(M^\dagger)_{ij} := M^*_{ji}$, and we have denoted the only non-zero $L \times L$-block of the diagonal matrix $D$ by $D' := 2\Gamma T M$. By formula (16) of the steady state current in chains, we only need to determine the correlation function $\nu(p_1^2)$. Since $D'_{ij} = \delta_{ij}(\delta_{i1} + \delta_{iL}) 2\gamma m_i T_i$ and $\nu(p_1^2) \equiv \nu(pp^{\mathrm{T}})_{11}$, the previous formula yields

$$\nu_{L;T_1,T_L}\big(p_1^2\big) \;=\; \frac{\gamma m_1^2}{\pi} \left\{ T_1 m_1 \int \phi^2 \big| Z(\phi)^{-1}_{11} \big|^2 \mathrm{d}\phi \,+\, T_L m_L \int \phi^2 \big| Z(\phi)^{-1}_{1L} \big|^2 \mathrm{d}\phi \right\}, \tag{33}$$

where $Z(w)^{-1}_{ij} := (Z(w)^{-1})_{ij}$.

Now, if one sets $T_L = T_1$ so that the system is in equilibrium, then by the equipartition theorem, or alternatively by Lemma 2.3, we know that the integral on the right hand side equals

$$\nu_{L;T_1,T_1}\big(p_1^2\big) \;=\; m_1 T_1\,.$$

Hence, by expressing both $m_1 T_1$ and $\nu_{L;T_1,T_L}(p_1^2)$ in terms of integral representation (33) in the general formula (16) of the steady state heat current in chains of oscillators, yields

$$\begin{aligned} \mathcal{J}_L(T_1, T_L) \;&=\; \frac{\gamma}{m_1} \Big\{ m_1 T_1 \,-\, \nu_{L;T_1,T_L}\big(p_1^2\big) \Big\} \\ &=\; \frac{\gamma}{m_1} \Big\{ \nu_{L;T_1,T_1}\big(p_1^2\big) \,-\, \nu_{L;T_1,T_L}\big(p_1^2\big) \Big\} \\ &=\; \frac{\gamma^2 m_1 m_L}{\pi} (T_1 - T_L) \int (2\pi w)^2 \big| Z(\phi)^{-1}_{1L} \big|^2 \mathrm{d}\phi\,. \end{aligned} \tag{34}$$

Now, by definition (31) we may write

$$Z(\phi)_{ij} \;=\; z_i(\phi)\delta_{ij} - \delta_{i,j+1} - \delta_{i,j-1}\,, \tag{35}$$

where $z_i \equiv z_i(\phi) = 2 + \mu - m_i\phi^2$ outside boundaries $2 \leq i \leq L-1$. Let us denote by $Z^{k,l}$ the $(L-1)\times(L-1)$-matrix obtained by removing the $k^{\text{th}}$ row and $l^{\text{th}}$ column from $Z$. Then a standard formula for matrices (see Lemma 2.4) yields the first equality of

$$Z(\phi)^{-1}_{1L} \;=\; (-1)^{L+1}\frac{\det Z(\phi)^{1,L}}{\det Z(\phi)} \;=\; \frac{1}{\det Z(\phi)}\,, \tag{36}$$

while the second one follows by using (35) to deduce $(-1)^{L+1}\det Z(\phi)^{1,L} = 1$.

We are now going to derive a recursion relation for the determinants $\det Z^{(n)}$ by using (35). Indeed, it follows by expanding the determinants w.r.t. the last column (or row) that

$$\begin{aligned}
\det Z^{(n)} \;&=\; z_n \cdot \det Z^{(n-1)} - \det Z^{(n-2)}\,,\\
\det Z^{(1)} \;&=\; z_1\,,\\
\det Z^{(0)} \;&=\; 1\,,
\end{aligned}$$

where the last initial condition follows since we have $\det Z^{(2)} = z_2 z_1 - 1$. Let us further write this recursion in terms of the transfer matrices

$$\begin{bmatrix}\det Z^{(n)}\\ \det Z^{(n)}\end{bmatrix} \;=\; \begin{bmatrix} z_n & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\det Z^{(n-1)}\\ \det Z^{(n-2)}\end{bmatrix}\,,$$

which holds for every $1 \leq n \leq L$ if one sets $\det Z^{(-1)} := 0$. Iterating then yields

$$\begin{aligned}
\det Z \;\equiv\; \det Z^{(L)} \;&=\; \begin{bmatrix}1 & 0\end{bmatrix}\begin{bmatrix} z_L & -1\\ 1 & 0\end{bmatrix} A_{L-1}\cdots A_2\begin{bmatrix} z_1 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}\\
&=\; \tilde{v}_L{}^{\mathrm{T}} A_L\, A_{L-1}\cdots A_2\, A_1\tilde{v}_1\,,
\end{aligned} \tag{37}$$

where we have introduced the new transfer matrices,

$$A_n \equiv A_n(\phi) := \begin{bmatrix} 2-\mu-\phi^2 m_n & -1\\ 0 & 1\end{bmatrix}\,, \tag{38}$$

which do not 'see' the boundaries, and transferred the boundary conditions into the vectors $\tilde{v}_1$ and $\tilde{v}_L$ such the last equality in (37) holds:

$$\tilde{v}_1(\phi) := \begin{bmatrix} 1\\ (2+\mu-\phi^2 m_1) - z_1\end{bmatrix}\,, \qquad \tilde{v}_L(\phi) := \begin{bmatrix} 1\\ z_L - (2+\mu-\phi^2 m_L)\end{bmatrix}\,. \tag{39}$$

By recalling definitions (35), (31) and (24), we obtain

$$\begin{aligned}
z_k^{(\mathrm{Free})} - (2+\mu-\phi^2 m_k) \;&=\; -1 + \mathrm{i}\phi\gamma m_k\,,\\
z_k^{(\mathrm{Fix})} - (2+\mu-\phi^2 m_k) \;&=\; \mathrm{i}\phi\gamma m_k\,,
\end{aligned} \tag{40}$$

for the boundary differences: $k = 1, L$ in (39). Plugging first (37) into (36) and then feeding the resulting expression into (34) yields a useful expression for the steady state current.

Proposition 2.1. *The stationary state heat current through the chain* (22) *is*

$$\mathcal{J}_L(T_1,T_L) = \; 2\,(T_1 - T_L)\int_0^\infty \left|v_L(\phi)^{\mathrm{T}} A_L(\phi)A_{L-1}(\phi)\cdots A_1(\phi)\,v_1(\phi)\right|^{-2}\mathrm{d}\phi\,, \tag{41}$$

*where transfer matrices $A_k$ only depend on the masses of the chain while the vectors*

$$v_k(\phi) := \pi^{1/4}(\gamma m_k|\phi|)^{-1/2}\tilde{v}_k(\phi)\,, \tag{42}$$

*take into account both the boundary conditions and the bath properties.*

Note that by using evenness of the functions w.r.t. $\phi$ we have limited the integral to positive values. Choosing fixed boundary conditions yield

$$v_1^{(\mathrm{Fix})}(\phi) := \pi^{1/4} \begin{bmatrix} (\gamma m_1 |\phi|)^{-1/2} \\ \mathrm{i}\,(\gamma m_1 |\phi|)^{1/2} \end{bmatrix} \qquad \text{and} \qquad v_L^{(\mathrm{Fix})}(\phi) := \pi^{1/4} \begin{bmatrix} (\gamma m_L |\phi|)^{-1/2} \\ -\mathrm{i}\,(\gamma m_L |\phi|)^{1/2} \end{bmatrix}, \quad (43)$$

which correspond to the coupling of the Casher-Lebowitz model (see [8]).

LEMMA 2.2. *Recall definitions* (23) *and* (31) *of matrices* $K$ *and* $Z(w)$*, respectively. Then:*

(i) *The zeros of* $\phi \mapsto \det Z(\phi) : \mathbb{C} \mapsto \mathbb{C}^{L\times L}$*, and the eigenvalues of* $K$ *are related by the complex rotation* $z \mapsto -\mathrm{i}z$*, i.e.,* $\det Z(\phi_0) = 0$ *if and only if* $\det(-\mathrm{i}\phi_0 I - K) = 0$*;*

(ii) *If* $\alpha \in \sigma(K)$ *then* $\Re\alpha > 0$ [7]*;*

(iii) *The exponential damping* $\|\exp(-tK)\| \leq C\exp(-ct)$ *applies for* $t \geq 0$ [8]*.*

PROOF. For proving (i), assume that $\alpha \in \sigma(K)$, so that $K\mathrm{u} = \alpha\mathrm{u}$ for some non-zero $\mathrm{u}^{\mathrm{T}} = [\mathrm{q}^{\mathrm{T}}\,\mathrm{p}^{\mathrm{T}}]$. By definition of $K$, this is equivalent to

$$\mathrm{p} = -\alpha M\mathrm{q}\,, \qquad \text{and} \qquad \Phi\mathrm{q} + \Gamma\mathrm{p} = \alpha\mathrm{p}\,. \quad (44)$$

Now $\mathrm{q} \neq 0$ since otherwise we would have also $\mathrm{p} = 0$ by the first equation. Combining these two equations yields

$$(\Phi + \alpha^2 M - \alpha\Gamma M)\,\mathrm{q} \;=\; 0\,. \quad (45)$$

But the right side equals $Z(\mathrm{i}\alpha)\mathrm{q}$, and hence it follows that $\det Z(\mathrm{i}\alpha) = 0$, when $\alpha \in \sigma(K)$. In the other direction, if $\det Z(\mathrm{i}\alpha) = 0$ then $Z(\mathrm{i}\alpha)\mathrm{q} = 0$, for $\mathrm{q} \neq 0$, and hence we may define $\mathrm{p} := -\alpha M\mathrm{q}$ so that both equations in (44) apply and hence $\alpha \in \sigma(K)$.

In order to prove (ii) suppose $K\mathrm{u} = \alpha\mathrm{u}$ for $\mathrm{u} \neq 0$, and apply $\mathrm{q}^{\mathrm{T}}$ from the left on both sides of (45) which must hold by part (i) of the lemma. This results in a quadratic polynomial of $\alpha$:

$$w\alpha^2 - v\alpha + u \;:=\; \mathrm{q}^{\mathrm{T}} M\mathrm{q}\,\alpha^2 - \mathrm{q}^{\mathrm{T}}\Gamma M\mathrm{q}\,\alpha + \mathrm{q}^{\mathrm{T}}\Phi\,\mathrm{q} \;=\; 0\,.$$

We will show that one can not have $q_1 = q_L = 0$, and then conclude the result since the roots of the resulting polynomial have the property $\Re\alpha > 0$. Suppose now that $q_1 = q_L = 0$, so that (45) reduces to $M^{-1}\Phi\mathrm{q} = -\alpha^2\mathrm{q}$. By using the definitions of $\Phi$ and $M$ one sees that $q_1 = \cdots = q_k = 0$ implies that also $q_{k+1} = 0$, $1 \leq k \leq L-1$, so that we must actually have $\mathrm{q} = 0$. But (44) then results in $\mathrm{p} = 0$ as well, contradicting the assumption $\mathrm{u} \neq 0$.

Part (iii) follows by putting $K$ into a Jordan form, and expressing it in a basis where $K = K' + L$ and $K'$ is a diagonal matrix containing the eigenvalues of $K$, and $L$ is nilpotent of order $2L-1$ at most. □

LEMMA 2.3. *For any* $\gamma > 0$ *we have the identity:*

$$\frac{1}{\pi}\int_{-\infty}^{\infty} \phi^2 Z(\phi)^{-1} M\Gamma (Z(\phi)^{-1})^{\dagger}\,\mathrm{d}\phi \;=\; M^{-1}\,. \quad (46)$$

[7] Notations: $z = \Re z + \mathrm{i}\Im z$, with $\Re z, \Im z \in \mathbb{R}$, for $z \in \mathbb{C}$.

[8] Here $\|\bullet\|$ can be chosen to be the operator norm induced by the Euclidean distance on $\mathbb{R}^L$.

PROOF. By definition of $Z(\phi)$, $Z(\phi)^\dagger = Z(-\phi)$, and

$$\phi M\Gamma \;=\; \Im\big[Z(\phi)\big] \;=\; \frac{Z(\phi)-Z(\phi)^*}{2\mathrm{i}} \;=\; \frac{Z(\phi)-Z(-\phi)}{2\mathrm{i}}\,.$$

Using these yields

$$\begin{aligned}\frac{1}{\pi}\int_{-\infty}^{\infty}\phi^2 Z(\phi)^{-1}M\Gamma Z(\phi)^{-1}\,\mathrm{d}\phi \;&=\; \frac{1}{\pi}\int_{-\infty}^{\infty}\phi Z(\phi)^{-1}\frac{Z(\phi)-Z(-\phi)}{2\mathrm{i}}Z(\phi)^{-1}\,\mathrm{d}\phi\\ &=\; \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\phi\left\{Z(\phi)^{-1}-(Z(\phi)^{-1})^*\right\}\mathrm{d}\phi \qquad (47)\\ &=\; \frac{1}{\pi}\Im\left\{\int_{-\infty}^{\infty}\phi Z(\phi)^{-1}\,\mathrm{d}\phi\right\}.\end{aligned}$$

We use Cauchy formula to evaluate the integral. First we note that $\phi \mapsto Z(\phi)^{-1}_{ij}$, $1 \le i,j \le L$, is a ratio of complex polynomials, and hence a holomorphic function. Moreover, by Lemma 2.2 we know that $Z(\phi) = 0$ if and only if $\Im\phi > 0$. Hence, by Cauchy formula:

$$\int_\gamma \phi\, Z(\phi)^{-1}\mathrm{d}\phi \;=\; 0\,, \qquad (48)$$

for any loop $\gamma : [0,1] \to \bar{\mathbb{H}}_- := \big\{z \in \mathbb{C} : \Im(z) \le 0\big\}$. By defining two curves $\gamma'_R(\tau) := 2R(\tau - 1/2)$ and $\gamma''_R(\tau) = R\mathrm{e}^{-\mathrm{i}\pi\tau}$ for $\tau \in [0,1]$ and $R > 0$, we see that (48) applied to the combined loop $\gamma_R := \gamma'_R * \gamma''_R$, gives

$$\int_{-\infty}^{\infty}\phi Z(\phi)^{-1}\mathrm{d}\phi \;=\; \lim_{R\to\infty}\int_{\gamma'_R}\phi Z(\phi)^{-1}\mathrm{d}\phi \;=\; -\lim_{R\to\infty}\int_{\gamma''_R} Z(\phi)^{-1}\mathrm{d}\phi\,. \qquad (49)$$

The integrand in the last expression is

$$Z(R\mathrm{e}^{\mathrm{i}\phi})^{-1} \;=\; -\big(I - \mathrm{i}R^{-1}\mathrm{e}^{\mathrm{i}\pi\tau}\Gamma - R^{-2}\mathrm{e}^{\mathrm{i}2\pi\tau}M^{-1}\Phi\big)^{-1}\frac{\mathrm{e}^{\mathrm{i}2\pi\tau}}{R^2}\,M^{-1} \;=\; (I+\mathcal{O}(1/R))\frac{\mathrm{e}^{\mathrm{i}2\pi\tau}}{R^2}\,M^{-1}\,.$$

Substituting this into the path integral (49) and then plugging the result into (47) yields the desired identity. □

LEMMA 2.4. *Suppose $B$ is a $n \times n$-matrix that has an inverse, and denote by $B^{i,j}$ the $(n-1)\times(n-1)$-matrix obtained by removing the $i^{th}$ row and the $j^{th}$ column of matrix $B$. Then*

$$(B^{-1})_{ij} \;=\; (-1)^{i+j}\frac{\det B^{i,j}}{\det B}\,.$$

PROOF. This basic result can be found for example in book [16]. □

## 2.6 Our results

By analyzing the evolution of the elements of the discrete time stochastic process

$$n \mapsto Q_n(\phi) := A_n(\phi)A_{n-1}(\phi)\cdots A_1(\phi)\,, \qquad (50)$$

for fixed $\phi$, we show in paper (A) that the averaged integrand in (41) satisfies the following scaling law.

THEOREM 2.5. *Consider a system* (22) *without the pinning potential* ($\mu = 0$) [9] *and fixed boundary conditions, so that* $v_1, v_L$ *are given by* (43). *Assume that the masses* $m_i$ *are i.i.d. and have Lebesgue-density* $\rho$ *which is compactly supported in* $(0, \infty)$, *continuously differentiable inside its support, with an uniformly bounded derivative. Then*

$$\begin{aligned} \phi^2 \cdot \mathbf{1}\{1 \le \phi^2 L \le 2\} &\lesssim \mathbb{E}\, \frac{1}{|v_L(\phi)^{\mathrm{T}} Q_L(\phi)\, v_1(\phi)|^2} \\ &\lesssim \max\bigl\{\phi L^{-1/2}, \phi^2\bigr\} \cdot \exp(-\gamma \phi^2 L)\,. \end{aligned} \tag{51}$$

The choice of $v_1, v_L$ corresponds to the so-called Casher-Lebowitz model (see [8]). By using (51) in (41) and changing the order of integrals, one confirms the long conjectured scaling behavior of the heat current in the Casher-Lebowitz model:

$$\mathbb{E}\, \mathcal{J}_L(T_1, T_L) \;\sim\; (T_1 - T_L) \cdot L^{-3/2}\,.$$

The upper bound in (51) had already been proven for the frequency regime $\phi \ge \varepsilon$, where $\varepsilon > 0$ is any constant independent of $L$ and $\phi$, by O'Connor [23].

We also reproduce a similar scaling result for another related model where the Ornstein-Uhlenbeck heat baths are replaced by semi-infinite ordered harmonic chains started from equilibrium Gibbs states corresponding to temperatures $T_1$ and $T_L$, respectively. In this model the steady state heat current is shown to scale like $L^{-1/2}$ (in an annealed sense) as was first proven by Verheggen in [31], who approached the problem by studying wave propagation in random media. Our analysis reveals that this particular model is special because some difficult oscillations cancel each others from the final expression for the current density. In particular, the dramatically different scaling of $\mathbb{E}\, \mathcal{J}_L(T_1, T_L)$ between these two models can be understood from the general formula (21) of harmonic systems. Compared to the Casher-Lebowitz model the heat baths of the Verheggen-model distribute much larger portion of their energy on the low frequency eigenmodes $\phi_\alpha \lesssim L^{-1/2}$ which have localization lengths $\ell_\alpha \gtrsim L$ exceeding the length of the chain.

Our representation of the random matrix $Q_L(\phi)$ yields an asymptotically accurate representation for the eigenvectors $\mathrm{q}_\alpha$ corresponding to the frequencies $\phi_\alpha \lesssim L^{-\epsilon}$, $\epsilon > 0$, as $L$ is increased. The matrix representation also provides exact formulas for the whole eigensystem $\{(\phi_\alpha, \mathrm{q}_\alpha) : 1 \le \alpha \le L^d\}$ when the system is ordered.

The assumption that the masses form an i.i.d. sequence is not essential to the proof of (51). With little extra effort one could consider a stationary ergodic sequence of masses instead. Although the quenched case can be conveniently approached by using the representation of paper (A), considering the scaling of quenched stationary heat current w.r.t. the measure of the random masses would require some more work.

[9] When $\mu > 0$ it can be shown recursively from (50) that $\|Q_L(\phi)\| \le \exp(-\gamma L)$ for all $\phi > 0$.

# 3 Diffusion of random walks in random environments

In the previous section we saw that even the simplest integrable Hamiltonian models of heat transport become extremely challenging in the presence of disorder when spatial dimension is higher than one. In this section we simplify the microscopic model of transport even further. Namely, we abandon Hamiltonian dynamics altogether and simply suppose that a conserved quantity, say energy, is carried by $N \sim L^d$ i.i.d. random walkers $X^{(\alpha)}$, $1 \leq \alpha \leq N$, on $\mathbb{R}^d$. This yields a model of a stochastic fluid which one may associate to empirical measures on $\mathbb{R}^d$ analogous to (10):

$$\tau_L(\mathrm{d}x\,\mathrm{d}t) \;=\; L^{-d}\sum_{\alpha=1}^{\rho L^d} \delta_{\{L^{-1}X^{(\alpha)}_{\lfloor L^2 t\rfloor}\}}(\mathrm{d}x)\,\mathrm{d}t\,,$$

where $\lfloor r\rfloor := \max\{m \in \mathbb{Z} : m \leq r\}$ is the floor of $r$. It is not difficult to show that in the hydrodynamic limit this measure has a density $T(x,t)$ which satisfies a linear heat equation (cf. [20]).

This solvable model of transport is made disordered by letting the common transition probabilities of the random walkers be correlated random variables. Since $X^{(\alpha)}$ are i.i.d., the problem of whether the empirical measure satisfies again a heat equation in the hydrodynamic scaling limit, reduces to the problem of whether a single walker diffuses. This brings us to the domain of a branch of probability theory commonly known as *random walks in random environments* (RWRE) [10]. Again, it turns out that the added disorder makes the model much more challenging to analyze than its ordered counterpart. Nevertheless, this time we will be able to treat a multidimensional case and show that the qualitative behavior of the model remains unchanged, provided the disorder is not too strong.

Besides providing the necessary definitions and stating some important results related to the diffusion of RWRE, we define a renormalization group (RG) transform acting on collections of transition probabilities. After providing some basic intuition behind the RG approach, we introduce an alternative way to look at the main proofs of paper (B). The section is concluded by a short discussion concerning two recent papers where the RG theory developed for RWRE is applied on more complicated models of transport. In this last part, some physical motivation why one should consider random environments with time-dependent disorder is also given.

## 3.1 Basic definitions and known scaling limits

Let $\mathbb{X}$ stand for either $\mathbb{Z}^d$ or $\mathbb{R}^d$. Suppose we have a probability measure $q_k(x)$ on $\mathbb{X}$ attached to each spatial $x \in \mathbb{X}$ and temporal coordinate $k \in \mathbb{N}_0$, respectively. The sequence of measures

$$\boldsymbol{q} \;=\; \big(q_k(x) : (k,x) \in \mathbb{N}_0 \times \mathbb{X}\big)\,, \tag{52}$$

defines uniquely the law $\mathsf{P}^{\boldsymbol{q}}$ of a discrete time random walk $X \equiv (X_k)$ on $\mathbb{X}$ by the usual Markov construction:

$$\mathsf{P}^{\boldsymbol{q}}\big[X_0 = 0\big] \;=\; 1\,, \qquad \text{and} \qquad \mathsf{P}^{\boldsymbol{q}}\big[X_{k+1} \in \mathrm{d}y \big| X_k = x\big] \;=\; q_k(x;\mathrm{d}y)\,. \tag{53}$$

This motivates us to consider $\boldsymbol{q}$ as an environment of random walks.

[10] For general references on RWRE, see [29], [33] or [32].

DEFINITION 3.1. *Let $\mathcal{P}(S)$ be the Borel-probability measures on a metric space $S$. An environment $\boldsymbol{q}$ for random walks on $\mathbb{X}$ is a collection* (52) *of probability measures $q_k(x) \in \mathcal{P}(\mathbb{X})$ associated to spacetime points $(k,x) \in \mathbb{N}_0 \times \mathbb{X}$. The set of all such environments is denoted by $\mathcal{E} \equiv \mathcal{E}(\mathbb{X})$. We use the following conventions:*

(i) *If $\boldsymbol{q} \in \mathcal{E}(\mathbb{R}^d)$ and $q_k(x)$ has a density w.r.t. Lebesgue measure we denote this density by $q_k(x,y)$ so that $q_k(x;A) = \int_A q_k(x,y)\mathrm{d}y$, for Borel sets $A \subset \mathbb{R}^d$;*

(ii) *If $\boldsymbol{q} \in \mathcal{E}(\mathbb{Z}^d)$ then $q_k(x,y)$ denotes the probability $q_k(x;\{y\})$ instead.*

*An environment $\boldsymbol{q} \in \mathcal{E}(\mathbb{X})$ induces the path measure $\mathsf{P}^{\boldsymbol{q}} \in \mathcal{P}(\mathbb{X}^{\mathbb{N}_0})$ for a random walk $X \equiv (X_k)$ on $\mathbb{X}$ through the Markov rules* (53)*, where in case $\mathbb{X} = \mathbb{Z}^d$, $q_k(x;\mathrm{d}y)$ is replaced by $q_k(x,y)$ in the second expression.*

The reason why we consider environments on $\mathbb{R}^d$ as well as $\mathbb{Z}^d$, as it has been customary, is that the renormalization transform, to be defined in the next subsection, is more naturally formulated in the continuous setting. Moreover, $\mathcal{E}(\mathbb{Z}^d)$ is actually included in $\mathcal{E}(\mathbb{R}^d)$: a given discrete environment $\boldsymbol{q} \in \mathcal{E}(\mathbb{Z}^d)$ can be interpreted as a continuous one $\tilde{\boldsymbol{q}} \in \mathcal{E}(\mathbb{R}^d)$ with uniformly bounded densities

$$\tilde{q}_k(x,y) \;=\; q_k(\lfloor x \rfloor, \lfloor y \rfloor)\,, \tag{54}$$

where $\lfloor x \rfloor \in \mathbb{Z}^d$ denotes the floor applied componentwise to $x$. The environments $\boldsymbol{q}$ and $\tilde{\boldsymbol{q}}$ are essentially equivalent: if $\tilde{X} \sim \mathsf{P}^{\tilde{\boldsymbol{q}}}$ then a walk $X \sim \mathsf{P}^{\boldsymbol{q}}$ is obtained by setting $X_k := \lfloor \tilde{X}_k \rfloor$. In particular, $X$ and $\tilde{X}$ have the same scaling behavior.

Suppose $(\Omega, \Sigma, \mathbb{P})$ is a probability space. Then a measurable function $\boldsymbol{Q} : \Omega \to \mathcal{E}$ is a *random environment*. Here $\mathcal{E}(\mathbb{Z}^d)$ is made measurable by augmenting it with the $\sigma$-algebra generated by the projections $\boldsymbol{q} \mapsto q_k(x,y)$. When $\mathbb{X} = \mathbb{R}^d$, the uncountable number of coordinates $x$ poses no problems either, as we will deal only with very regular random environments, e.g., those derived through (54) from discrete random environments.

The joint law of the pair $(X, \boldsymbol{Q})$, $X \sim \mathsf{P}^{\boldsymbol{Q}}$, is then a probability measure $\mu$ on $\mathcal{E} \times \mathbb{X}^{\mathbb{N}_0}$, which has the skew structure

$$\mu(\mathrm{d}\boldsymbol{q}, \mathrm{d}\underline{x}) \;=\; \mathbb{P}(\boldsymbol{Q} \in \mathrm{d}\boldsymbol{q})\, \mathsf{P}^{\boldsymbol{q}}(\mathrm{d}\underline{x})\,, \tag{55}$$

where $\underline{x} \equiv (x_k : k \in \mathbb{N}_0) \in \mathbb{X}^{\mathbb{N}_0}$. Adapting the terminology commonly used with disordered systems, the path measure $\mathsf{P}^{\boldsymbol{Q}(\omega)}$ corresponding to a fixed realization $\boldsymbol{Q}(\omega)$ of the environment is called the *quenched* measure (of the random walk). Similarly, the *annealed measure* $\bar{\mathsf{P}}$ (of the walker) is generated by averaging over the quenched environments:

$$\bar{\mathsf{P}}(\mathrm{d}\underline{x}) \;:=\; \mathbb{E}\,\mathsf{P}^{\boldsymbol{Q}}(\mathrm{d}\underline{x}) \;\equiv\; \int \mathsf{P}^{\boldsymbol{Q}(\omega)}(\mathrm{d}\underline{x})\,\mathbb{P}(\mathrm{d}\omega)\,.$$

A process $(\bar{X}_k)_{k\in\mathbb{N}_0}$ evolving with respect to the annealed measure is generally not Markov, since the past coordinates on its path contain information about the medium which must be taken into account when averaging with respect to the environments. Indeed, it can sometimes be fruitful to approach a non-Markovian process $Y$ by attempting to construct an RWRE where the annealed law equals to that of $Y$.

As with disordered systems in general, one usually translates the symmetries of the ordered model into the law of the disorder. In particular, the following are frequently assumed.

DEFINITION 3.2. *A random environment* $\boldsymbol{Q} = (Q_k(x) : (k,x) \in \mathbb{N}_0 \times \mathbb{X}) : \Omega \to \mathcal{E}$ *is said to be*

(i) *static or frozen, if it does not change in time, i.e.,* $Q_k(x) = Q_0(x) =: Q(x)$ *for all pairs* $(k,x)$ $\mathbb{P}$*-almost surely. Otherwise,* $\boldsymbol{Q}$ *is said to be time-dependent or dynamic;*

(ii) *statistically translation invariant, provided it has the same law as the spacetime shifted environment* $\theta_{k_0,x_0}\boldsymbol{Q}$*, with components* $(\theta_{k_0,x_0}Q)_k(x) := Q_{k+k_0}(x+x_0; \bullet + x_0)$*, for any* $k \in \mathbb{N}_0$ *and* $x_0 \in \mathbb{X}$*;*

(iii) *statistically isotropic, provided the law of* $\boldsymbol{Q}$ *is invariant under rotations which fix* $\mathbb{Z}^d$ [11].

The next result by Sinai [26] shows that disorder in general has dramatic effects in static one-dimensional settings.

THEOREM 3.3. *Consider a static* $\boldsymbol{Q} : \Omega \to \mathcal{E}(\mathbb{Z})$*. Assume that the components* $Q(x) \equiv Q_k(x)$ *are i.i.d., simple, i.e.,*

$$Q^\omega(0,-1) \,+\, Q^\omega(0,+1) \;=\; 1\,, \qquad \mathbb{P}-a.e.\ \omega\,,$$

*and uniformly elliptic in the sense that for some* $0 < \delta < 1$*,*

$$0 < \delta \;\le\; Q^\omega(0,\pm 1) \;\le\; 1-\delta\,, \qquad \mathbb{P}-a.e.\ \omega\,.$$

*Suppose moreover that the walk is statistically balanced and non-deterministic, i.e.,*

$$\mathbb{E}\,\ln Q(0,1) = \mathbb{E}\,\ln Q(0,-1)\,, \qquad \textit{and} \qquad \mathbb{P}\big\{Q(0,1) = 1/2\big\} < \,1\,.$$

*Then there exists a function* $Z_n : \mathcal{E}(\mathbb{Z}) \to \mathbb{R}$ *such that for any* $\eta > 0$*,*

$$\lim_{n\to\infty} \mu\left\{ (\boldsymbol{q}, \underline{x}) \,:\, \Big|\frac{\sigma^2 x_n}{(\ln n)^2} - Z_n(\boldsymbol{q})\Big| \;\ge\; \eta \right\} \;=\; 0\,, \quad \textit{where} \quad \sigma^2 \,:=\, \mathbb{E}\,\ln^2 \frac{Q(0,1)}{Q(0,-1)}\,,$$

*and the joint law* $\mu$ *is defined in* (55)*. Moreover,* $Z_n$ *has a limit distribution* $G$*, i.e.,*

$$\lim_{n\to\infty} \mathbb{P}\big\{Z_n \le z\big\} = \; G(z)\,.$$

This seminal result known as *Sinai's localization* says that unless a statistically balanced environment is completely non-random, the environment acts effectively as a huge *trap*, or a potential valley, which prevents the particle from diffusing normally. The key mechanism behind this perhaps surprising results is not difficult to understand. Indeed, consider the random potential

$$V(y) \;:=\; \sum_{x=0}^{y} U(x)\,, \qquad \text{where} \quad U(x) := \ln \frac{Q(x,x-1)}{Q(x,x+1)}\,,$$

where $U(x)$, $x \in \mathbb{Z}$, are by assumptions i.i.d. zero mean random variables. Hence, typically

$$\max_{0\le x,y\le D} (V(y) - V(x)) \;\sim\; D^{1/2}\,,$$

[11] It would be perhaps more logical to use this definition only when $\mathbb{X} = \mathbb{Z}^d$ and demand full rotational invariance when $\mathbb{X} = \mathbb{R}^d$. However, for our purposes the current definition is more appropriate.

for $D \in \mathbb{N}$. This implies that for the random walk to move the distance $D$ from the origin to the right, it has to cross a potential barrier of size $\mathcal{O}(\exp(D^{1/2}))$. But getting over such barrier corresponds to taking $\exp(D^{1/2})$ steps against a drift of order $\mathcal{O}(1)$ in a row, and hence it will take at least time $n \sim \exp(D^{1/2})$ for such an unlikely event to take place. Of course, the same argument applies to the other direction, and hence one concludes that in time $n$ the walker is expected to reach a distance of order $(\ln n)^2$ from the origin.

The main lesson from Sinai's result is that the existence of traps should be taken seriously. In higher dimensions, there are more 'directions' for a walker to escape traps, and therefore one expects phase transitions to take place w.r.t. the scaling behavior of the walker as the strength of the disorder, and hence seriousness of the traps, is varied. It has been further shown that when $\mathbb{E} \ln U(0) \neq 0$, then the sign of this quantity also determines whether the walk in transient to left or right (cf. [27]). Moreover, these type of results have been generalized for ergodic environments with more general jumps, etc.

Letting the environment become a dynamic stationary process favors diffusive behavior in general as well. Namely, any serious trap where the walker might find itself stuck is expected to be wiped out by the time evolution before the walker would have found its way out of the corresponding frozen trap by itself. It is perhaps worth clarifying here that although arbitrary bad traps will form almost surely, the one particular trap which the walker has 'selected' is still a large deviation w.r.t. the law of the stationary environment.

Another, way to see why the dynamics should weaken the trapping effects is to interpret the original random walk $X$ on a dynamic environment on $\mathbb{Z}^d$ as a random walk $\widetilde{X}$ on the $(d+1)$-dimensional static environment $\mathbb{Z} \times \mathbb{Z}^d$ with a constant drift,

$$\tilde{X}_k \,:=\, (\tilde{X}^0_k, X^1_k, \ldots, X^d_k) \,\equiv\, (k, X^1_k, \ldots, X^d_k)\,, \qquad \tilde{Q}(\tilde{x}, \tilde{y}) \,:=\, \delta_{\tilde{x}^0+1, \tilde{y}^0}\, Q_{\tilde{x}^0}(x, y)\,, \quad (56)$$

so that $X \sim \mathsf{P}^{\boldsymbol{Q}}$ implies $\widetilde{X} \sim \mathsf{P}^{\widetilde{\boldsymbol{Q}}}$. As increasing dimensions and adding drift both help the walker to escape traps, we conclude that dynamics should indeed favor diffusive behavior.

The strongest general results concerning dynamic environments in higher dimensions have assume independence of $Q_k = (Q_k(x) : x \in \mathbb{X})$. Although, not anymore [12] the most general one we give the following theorem in [24] as an example.

THEOREM 3.4. *Let $\boldsymbol{Q} : \Omega \to \mathcal{E}(\mathbb{Z}^d)$ and assume that $Q_k(x)$'s are i.i.d. Furthermore, suppose that the steps are not deterministic and their variance is bounded, i.e.,*

$$\mathbb{P}\big\{\sup_y Q_0(0,y) < 1\big\} > \,0\,, \qquad \Gamma \,:=\, \textstyle\sum_y (y-v)(y-v)^{\mathrm{T}} \mathbb{E}\, Q_0(0,y) \,\in\, \mathbb{R}^{d\times d}\,.$$

*Let $X \sim \mathsf{P}^{\boldsymbol{Q}}$ and define an RCLL[13] functions $X^{(l)} : [0,\infty) \to \mathbb{R}^d$, $l \in \mathbb{N}$, by centering and scaling $X$:*

$$X^{(l)}(t) \,:=\, \frac{X_{\lfloor lt \rfloor} - v\,\lfloor lt \rfloor}{\sqrt{l}}\,, \qquad v \,:=\, \sum_y y\, \mathbb{E}\, Q_0(0,y)\,. \quad (57)$$

*Let $W$ be a standard $d$-dimensional Brownian motion.*

[12] In [18] this result is generalized so that $Q_0(x)$ and $Q_0(y)$ may be correlated, as long as their covariance decays like $|x-y|^m$, with $m > 26$.

[13] RCLL stands for 'Right Continuous with Left Limits'.

*Then a quenched invariance principle,*

$$X^{(l)} \xrightarrow{w} \Gamma^{1/2} W\,, \qquad \text{as } l \to \infty \qquad \text{for } \mathbb{P}\text{-a.e. } \omega\,, \tag{58}$$

*applies in the weak sense in Skorohod topology of RCLL functions* $f : [0,T) \to \mathbb{R}^d$.

We also remark that the space-time i.i.d. environments considered in this theorem are very special in the sense that the annealed measure reduces the random walk into a regular spacetime homogenous random walk on $\mathbb{Z}^d$. Namely, although $\boldsymbol{q} \mapsto \mathsf{P}^{\boldsymbol{q}} : \mathcal{E} \to \mathcal{P}(\mathbb{X}^{\mathbb{N}_0})$ is nonlinear, we nevertheless have under the assumptions of the theorem

$$\mathbb{E}\,\mathsf{P}^{\boldsymbol{Q}} \;=\; \mathsf{P}^{\mathbb{E}\,\boldsymbol{Q}}\,, \tag{59}$$

where we have defined the *average environment* componentwise:

$$\mathbb{E}\,\boldsymbol{Q} \;:=\; (\mathbb{E}\,Q_k(x) : (k,x) \in \mathbb{N}_0 \times \mathbb{R}^d)\,. \tag{60}$$

Due to (56) we may assume without loosing generality that a given random environment $\boldsymbol{Q}$ on $\mathbb{Z}^d$ is static. The so-called *environment as seen by walker* (ESW) process is a Markov chain $(\boldsymbol{Q}_{(k)})_{k\in\mathbb{N}_0}$ on $\mathcal{E}(\mathbb{Z}^d)$ obtained by shifting $\boldsymbol{Q}$ according to the position of the walker $X \sim \mathsf{P}^{\boldsymbol{Q}}$,

$$\boldsymbol{Q}_{(k)} \;:=\; \theta_{0,X_k}\boldsymbol{Q}\,, \qquad k \in \mathbb{N}_0\,, \tag{61}$$

so that $\boldsymbol{Q}_{(k)} = (Q_{(k)}(x) : x \in \mathbb{Z}^d)$ with $Q_{(k)}(x,y) = Q(x + X_k, y + X_k)$. Whenever the invariant measure of this chain is absolutely continuous with respect to the law of $\boldsymbol{Q}$, then laws of large numbers and CLTs can be deduced (cf. [21]). A quite satisfactory theory for static RWRE in one-dimension exists since the invariant measure can be constructed explicitly in that setting. In higher dimensions, the general lack of sufficient estimates for the invariant measure of the ESW process makes the analysis much more challenging. In some special cases where an additional balancing condition holds, ESW approaches have been efficient (cf. [22]). More recently, ESW ideas have been applied successfully for dynamic environments (cf. [18] and references therein). For example, Theorem 3.4 is proven using such approach.

In general multidimensional setting, assuming more complicated spacetime correlations between the elements $Q_k(x,y)$ leaves one with the perturbative results, where the disorder $Q_k - \mathbb{E}\,Q_k$ must be assumed to be sufficiently weak. The most general perturbative results include the works [11], [12] and [6] which consider Markovian environments, deterministic but strongly chaotic environments, and environments where the correlations decay exponentially in both space and time, respectively.

In the multidimensional setting where $\boldsymbol{Q}$ is static and not assumed to have 'atypical' properties which reduce the effects of traps, the most general perturbative result is the following proven in [5].

THEOREM 3.5. *Suppose* $\boldsymbol{Q} : \Omega \to \mathcal{E}(\mathbb{Z}^d)$, $d \geq 3$, *is static, statistically translation invariant and isotropic, with i.i.d. components* $Q(x) \equiv Q_k(x)$, $x \in \mathbb{Z}^d$. [14] *Moreover, assume that the jump lengths have almost surely uniformly bounded exponential tails,*

$$\sup_y \mathrm{e}^{-|y|} Q(0,y) \;\leq\; C_0\,,$$

[14] The assumptions stated here are not in their most general form (see: page 350 of [5]).

*for some $C_0 > 0$, and the following probabilistic estimates*

$$\begin{aligned} \mathbb{E}\exp[t\,Q(0,y)] &\le \exp\big[t(\mathbb{E}\,Q(0,y)+\varepsilon^2)\big]\,, \quad t\in[0,\infty)\,,\\ \mathbb{P}\big\{Q(0,y) > 1-\mathrm{e}^{-m}\big\} &\le \exp(-\gamma m)\,, \qquad\qquad m\in\mathbb{N}\,, \end{aligned}$$

*apply for $\varepsilon, \gamma > 0$.*

*Then there exist threshold values $\varepsilon_+, \gamma_- > 0$ such that, if $\varepsilon \le \varepsilon_-$ and $\gamma \ge \gamma_+$ then process $X \sim \mathsf{P}^{\boldsymbol{Q}}$ satisfies an almost sure invariance principle, i.e., the scaled processes $X^{(l)}$ defined by (57) with $v = 0$, satisfy the almost sure weak limit (58) with a diagonal correlation matrix $\Gamma_{ij} = \sigma\delta_{ij}$.*

This result is proven by using the renormalization group formalism. It has been reproduced by a different multiscale method in continuous space-time setting in [30]. For the rest of this section we will concentrate on the renormalization techniques derived from the ideas introduced originally in the prove of this seminal result.

## 3.2 Renormalization

Suppose that we want to prove that random walks in an environment diffuse. This can be sometimes accomplished by studying how the following renormalization group transform acts on it.

DEFINITION 3.6. *Let $\mathcal{E} \equiv \mathcal{E}(\mathbb{R}^d)$. The Gaussian renormalization group (RG) transform $\mathrm{RG}_l : \mathcal{E} \to \mathcal{E}$ with scaling parameter $l \in \mathbb{N}$, is defined by setting the components $q'_k(x)$ of the renormalized environment $\boldsymbol{q}' = \mathrm{RG}_l(\boldsymbol{q})$ to be so that* [15]

$$q'_k(x;\mathrm{d}y) \;=\; \mathsf{P}^{\boldsymbol{q}}\big[\, l^{-1/2}X_{l(k+1)} \in \mathrm{d}y \,\big|\, l^{-1/2}X_{lk} = x\,\big]\,, \quad \forall\, k,x,y\,. \tag{62}$$

It is clear that $\mathrm{RG}_l$ is non-linear. Moreover, it does not have an inverse since in (62) the components $X_k$, $k \notin l\mathbb{N}_0$, of the walker are integrated. However, the operators $\mathrm{RG} \equiv (\mathrm{RG}_l : l \in \mathbb{N})$ form a one-parameter (multiplicative) *semigroup* in the sense that

$$\mathrm{RG}_1 \;=\; \mathrm{Id}_{\mathcal{E}}\,, \qquad \mathrm{RG}_l \circ \mathrm{RG}_m \;=\; \mathrm{RG}_{lm}\,, \quad l,m\in\mathbb{N}\,.$$

Now, it is known that regular random walks satisfy under quite general conditions central limit theorems. This universal scaling behavior continues to hold for random walks in a rather large class of both dynamic and static inhomogenous environments $\boldsymbol{q} \in \mathcal{E}(\mathbb{R}^d)$ as is demonstrated by Theorem 3.4 and Theorem 3.5, respectively. The reason why such universal behavior takes place can be elegantly explained in terms of the map $\mathrm{RG}_l$. Namely, if $l \mapsto \mathrm{RG}_l(\boldsymbol{q}) : \mathbb{N} \to \mathcal{E}$ is considered as a dynamical system then an attracting fixed point $\boldsymbol{q}^* \in \mathcal{E}$ of the evolution map $\mathrm{RG}_1$ corresponds to an universal scaling limit.

Since we want to prove diffusion, we are interested in homogenous *Gaussian fixed points* $\mathrm{RG}_l(\boldsymbol{q}^*) = \boldsymbol{q}^*$, where $q^*_k(x,y) = C\exp\big[-\frac{1}{2}(x-y)^{\mathrm{T}}\Gamma^{-1}(x-y)\big]$ for each $(x,k)$, and the correlation matrix $\Gamma \in \mathbb{R}^{d\times d}$ is positive definite. All environments which converge to a such a Gaussian environment constitute the *Gaussian stable manifold* or *Gaussian domain of attraction* on $\mathcal{E}$:

$$\mathcal{G} \;:=\; \big\{\boldsymbol{q}\in\mathcal{E} : \exists\, \boldsymbol{q}^*\in\mathcal{E} \text{ s.t., } \lim\nolimits_{l\to\infty}\mathrm{RG}_l(\boldsymbol{q}) = \boldsymbol{q}^*\big\}\,, \tag{63}$$

[15] Note that in paper (B) the RG transform is defined with $l^2$ in place of $l$ for notational convenience.

where $\boldsymbol{q}^{(n)} \to \boldsymbol{q}$, for $\boldsymbol{q}^{(n)}, \boldsymbol{q} \in \mathcal{E}$, denotes weak convergence $q_k^{(n)}(x) \xrightarrow{w} q_k(x)$ for every $(k,x) \in \mathbb{N}_0 \times \mathbb{R}^d$. Since $\mathcal{G}$ can be parametrized by positive definite matrices $\Gamma \in \mathbb{R}^{d\times d}$ it follows that $\dim(\mathcal{G}) = (d+1)d/2$. When $\boldsymbol{q} \in \mathcal{G}$, a CLT holds by definition:

$$\mathsf{P}^{\boldsymbol{q}}\big[l^{-1/2}X_l \in \bullet\big] \;=\; \mathsf{P}^{\mathrm{RG}_l(\boldsymbol{q})}\big[X_1 \in \bullet\big] \;=\; (\mathrm{RG}_l(\boldsymbol{q}))_0(0) \;\xrightarrow{w}\; \mathrm{N}_\Gamma\,, \quad \text{as } l \to \infty\,. \tag{64}$$

Here, $\mathrm{N}_\Gamma$ is a centered Gaussian variable with correlation matrix $\Gamma$. The standard Lindeberg CLT (cf. page 90 in [19]) takes in RG-language the following form:

COROLLARY 3.7. *Let $\boldsymbol{q} \in \mathcal{E}$ be translation invariant in both space and time, i.e., suppose $q_k(x) = q \in \mathcal{P}(\mathbb{R}^d)$ for all $k, x$ with $\int x\, q(\mathrm{d}x) = 0$ and $\int x x^{\mathrm{T}} q(\mathrm{d}x) = \Gamma$ with a positive definite matrix $\Gamma$.*

*Then $\boldsymbol{q}_l := \mathrm{RG}_l(\boldsymbol{q})$ converges to a Gaussian fixed point with covariance matrix $\Gamma$ in the sense that $q_{l;k}(x) \to \mathrm{N}_\Gamma$ weakly as $l \to \infty$.*

Similarly, one could show that a functional CLT, or invariance principle applies as well. Moreover, if $\boldsymbol{q}$ develops a constant drift $v \in \mathbb{R}^d$ at large scales, the fluctuating part $X_k - vk$ can be studied in terms of the centered environment $\tilde{\boldsymbol{q}}$ with components $\tilde{q}_k(x) := q_k(x + vk; \bullet + vk)$.

Before considering the RG-transform on random environments, we note that RG-techniques are not restricted to diffusive nor Gaussian scaling limits. For example, for an environment $\boldsymbol{q}$ consisting of elements $q_k(x)$ with infinite first moments, one could try to show that the random walks start to resemble a stable Lévy-process under appropriate scaling $(x,k) \mapsto (l^a x, lk)$, $a \in (0,1)$, of space and time (cf. [14]). For such environments the RG-transform has to be modified so that on the right hand side of definition (62), instances of $l^{-1/2}X_{l\bullet}$ are replaced by $l^{-a}X_{l\bullet}$. With some minor modifications one could also consider environments in continuous time, etc.

## 3.3 Our results on weakly disordered random environments

In order to prove theorems using RG-transform we need to assume that the environments are drawn from a statistical ensemble with some simplifying symmetries. To avoid repetitions we will make the following standing assumptions on the random environments:

- *$\boldsymbol{Q} : \Omega \to \mathcal{E} \equiv \mathcal{E}(\mathbb{R}^d)$ is (statistically) translation invariant, and admits Lebesgue densities $Q_k(x,y)$ for every $(k,x)$ almost surely.*

Note that the assumption of the existence of densities is not a major restriction as we may study random environments on $\mathbb{Z}^d$ by using the association (54).

Besides the symmetries we also need to assume that $\boldsymbol{Q}$ is weakly disordered. To this end it is natural to split $\boldsymbol{Q}$ into its average $\mathbb{E}\,\boldsymbol{Q}$ (cf. (60)) and the remaining small random part

$$\boldsymbol{B} \;:=\; (B_k(x) : (k,x) \in \mathbb{N}_0 \times \mathbb{R}^d)\,, \qquad B_k(x) := Q_k(x) \,-\, \mathbb{E}\,Q_k(x)\,, \tag{65}$$

so that by summing componentwise one, has $\boldsymbol{Q} = \mathbb{E}\,\boldsymbol{Q} + \boldsymbol{B}$. In RG-parlance the question we seek to answer now reads as:

- Besides 'smallness', what needs to be assumed about $\boldsymbol{B}$ so that $\mathbb{P}\big\{\boldsymbol{Q} \in \mathcal{G}\big\} = 1$?

The picture we have in mind is simple: the translationally invariant deterministic part $\mathrm{RG}_l(\mathbb{E}\,\boldsymbol{Q})$ is attracted towards a Gaussian fixed point under very general assumptions as $l \to \infty$ (cf. Corollary 3.7). If the disorder $\boldsymbol{B}$ is weak enough, then it is not able to prevent this convergence to a Gaussian fixed point. However, it will be able to perturb the trajectory of the RG-flow so that the covariance matrix $\Gamma$ corresponding to $\mathbb{E}\,\boldsymbol{Q}$ gets *renormalized* [16] to a different value $\tilde{\Gamma}$ where $\tilde{\Gamma} - \Gamma$ is related to the 'size' $\boldsymbol{B}$ when the disorder is small.

In order to have a precise notion for the smallness of $\boldsymbol{B}$ we introduce a statistical distance of a (translationally invariant) random environment $\boldsymbol{Q}$ from its average. Since this will be done in terms of the cumulants of the kernels $Q_k(x,y)$ we need to introduce the following notations. Let $\mathbb{C}\mathrm{on}\{X_1,\dots,X_m\}$ stand for the cumulant of a set of random variables $X_i : \Omega \to \mathbb{C}$ w.r.t. the measure $\mathbb{P}$, and let us denote the function

$$(x_1,y_1;\dots;x_m,y_m) \;\mapsto\; \mathbb{C}\mathrm{on}\big\{Q_{k_1}(x_1,y_1),\dots,Q_{k_m}(x_m,y_m)\big\} \;:\; (\mathbb{R}^d\times\mathbb{R}^d)^m \to \mathbb{R}\,, \quad (66)$$

by $\mathbb{C}\mathrm{on}(Q_{k_1},\dots,Q_{k_m})$. A precise measure for the disorder of $\boldsymbol{Q}$ now reads as follows.

Definition 3.8. *Let $\Lambda_\varpi \subset \cup_{m\geq 2}\mathbb{N}_0^m$ be the set of all finite dimensional increasing vectors $(k_1,\dots,k_m)$ in which no more than $\varpi \in \mathbb{N}$ elements are allowed to be equal and $k_1 := 0$. The disorder $\mathbb{D}[\,\bullet\,]$ acting on random environments and parametrized by $\varpi$ and $\alpha,\beta,\gamma \geq 0$, is defined by*

$$\begin{aligned}\mathbb{D}[\boldsymbol{Q}] \;&\equiv\; \mathbb{D}_{\alpha,\beta,\gamma,\varpi}[\boldsymbol{Q}]\\ &:=\; \inf\left\{\varepsilon>0 : \sum_{(k_1,\dots,k_m)\in\Lambda_\varpi} \varepsilon^{-m}\prod_{i=1}^{m-1}(1+k_{i+1}-k_i)^\alpha \left\|\mathbb{C}\mathrm{on}(Q_{k_1},\dots,Q_{k_m})\right\|_{\beta,\gamma} \leq\, 1\right\},\end{aligned} \quad (67)$$

*where $\|\bullet\|_{\beta,\gamma}$ are norms acting on functions $g : (\mathbb{R}^d\times\mathbb{R}^d)^m \to \mathbb{C}$,*

$$\|g\|_{\beta,\gamma} := \sup_{k,l}\sup_{\pi\in S(m)}\sup_{x_{\pi(1)}}\int\cdots\sup_{x_{\pi(1)}}\int (1+|x_k-x_l|)^\beta\, \mathrm{e}^{\gamma\sum_i|x_i-y_i|}|g(z)|\,\mathrm{d}y_{\pi(m)}\cdots\mathrm{d}y_{\pi(1)}\,. \quad (68)$$

*Here, $S(m)$ denotes the permutations of the first $m$ integers, and we have written $z := (z_1,\dots,z_m)$ for the pairs $z_i := (x_i,y_i)$.*

The idea to quantify the strength of the disorder through cumulants in dynamic random environments was originally introduced in [6]. Recently in [25], this approach was refined by measuring the size of the cumulants with a norm close to (68). The main improvement of the present norm compared to the one used in [25] is that it covers spatial decorrelation of random environments, and consequently makes it possible to prove quenched results.

Before continuing further, let us make a few comments in order to clarify the meaning of the rather complex definition. As a first remark, we note that for a given $\boldsymbol{Q}$, the function $(\alpha,\beta,\gamma,\varpi) \mapsto \mathbb{D}_{\alpha,\beta,\gamma,\varpi}[\boldsymbol{Q}]$ is increasing in each of its argument. Moreover, the value $\mathbb{D}[\boldsymbol{Q}]$ does not depend on $\mathbb{E}\,\boldsymbol{Q}$, since vectors $(k_1,\dots,k_m) \in \Lambda_\varpi$ appearing in the

[16] Theorem 3.4 is again special in this sense: the independence of $Q_k$'s means that the correlation matrix does not change in the RG flow. This is easy to see from (59): Annealed and quenched CLTs have the same covariance matrix, and the former equals the one of average environment by (59).

sum have at least two components $m \geq 2$. Thus $\mathbb{D}[\boldsymbol{Q}]$ measures only the size of $\boldsymbol{B}$, and in a rather weak sense, as is demonstrated by the estimate

$$\sum_{k'} \iint \left| \mathbb{E}\, B_k(x,y) B_{k'}(x',y') \right| \mathrm{d}y \, \mathrm{d}y' \;\leq\; (\mathbb{D}_{0,0,0,\varpi}[\boldsymbol{Q}])^2 \,, \qquad \forall\, x, x' \,, \tag{69}$$

where $k'$ is summed over $\mathbb{N}_0 \backslash \{k\}$ if $\varpi = 1$, and otherwise over $\mathbb{N}_0$. As suggested by (69) when $\varpi = 1$, the equality $\mathbb{D}[\boldsymbol{Q}] = 0$ implies that $Q_k$ and $Q_{k'}$ are merely independent but may not vanish. Moreover, the bound (69) shows that $\mathbb{D}[\boldsymbol{Q}] < \infty$ only when the correlations between $Q_k$'s are integrable in time. By expressing moments in terms of the cumulants, similar bounds as (69) are obtained for higher moments of the disorder.

The meaning of the parameters $\alpha, \beta, \gamma \geq 0$ is best understood by replacing the function $B_k(x,y) B_{k'}(x',y')$ with

$$(1 + |k - k'|)^{\alpha} (1 + |x - x'|)^{\beta} \exp(\gamma |x - y| + \gamma |x' - y'|) \cdot B_k(x,y) B_{k'}(x',y') \,,$$

and $\mathbb{D}_{0,0,0,\varpi}[\boldsymbol{Q}]$ with $\mathbb{D}_{\alpha,\beta,\gamma,\varpi}[\boldsymbol{Q}]$ in (69).

Besides being a measure of disorder, the main attribute of $\mathbb{D}[\,\bullet\,]$ is that it contracts under renormalization transform. The following proposition is the main ingredient for the results of paper (B).

PROPOSITION 3.9. *Let $\boldsymbol{Q} : \Omega \to \mathcal{E}$, with the convolution powers $\bar{q}^{*l}$ of the average density*

$$\bar{q}(y) \;:=\; \mathbb{E}\, Q_0(0,y) \,,$$

*satisfying the following exponential bounds for every $l \in \mathbb{N}$ and $x \in \mathbb{R}^d$:*

$$\begin{aligned} \bar{q}^{*l}(x) \;&\leq\; K\, l^{-\frac{d}{2}}\, \mathrm{e}^{-\lambda l^{-1/2} |x|} \,, \\ \left| \bar{q}^{*l}(x + w) - \bar{q}^{*l}(x) \right| \;&\leq\; K\, l^{-\frac{d+1}{2}}\, \mathrm{e}^{-\lambda l^{-1/2} |x|} \,, \quad |w| \leq 1 \,. \end{aligned} \tag{70}$$

*Here, the constants $K$ and $\lambda > 0$ are independent of $l, x, w$. Let $\alpha, \beta, \gamma > 0$, and denote $\mathbb{D}[\,\bullet\,] \equiv \mathbb{D}_{\alpha,\beta,\gamma,\varpi}[\,\bullet\,]$.*

*Then there exist exponents $\mu, \nu > 0$, and an integer $\tilde{\ell}_\lambda(K) \geq 3$, such that if the disorder of $\boldsymbol{Q}$ is sufficiently weak in the sense that*

$$\mathbb{D}[\boldsymbol{Q}] \;\leq\; \ell^{-\mu} \,, \quad \textit{for some} \quad \ell \;\geq\; \tilde{\ell}_\lambda(K) \,, \tag{71}$$

*then it is even weaker for the renormalized environment $\boldsymbol{Q}' := \mathrm{RG}_\ell(\boldsymbol{Q})$:*

$$\mathbb{D}[\boldsymbol{Q}'] \;\leq\; \ell^{-\nu}\, \mathbb{D}[\boldsymbol{Q}] \,. \tag{72}$$

*Furthermore, the function $K \mapsto \tilde{\ell}_\lambda(K)$ is increasing.*

Note that estimates (70) are satisfied by normal densities with positive covariance matrices of size $\mathcal{O}(1)$. Moreover, the uniform bounds (70) could be replaced by weaker but more complicated integral bounds involving exponentially weighted $L^1$-norms.

While the proof of this result constitutes the bulk paper (B), it is not difficult to understand at the heuristic level why it is true. Namely, if $\ell$ is sufficiently large (cf. the second eq. of (71)) then one may use homogenization effects (guaranteed by the estimates (70)), such as averaging and smoothing, to show that the effective disorder becomes weaker at the larger scale $\ell$ up to the leading order in $\varepsilon := \mathbb{D}[\boldsymbol{Q}]$. Now, if $\varepsilon$ is sufficiently small (cf. the first eq. of (71)) then the contraction of the disorder

under this linearized RG map continues to hold true also after taking into account the effects of the non-linear terms. This possibility to essentially ignore the non-linear terms corresponds to the observation that the disorder is too weak to generate severe traps or other difficulties in $\ell$ steps. In conclusion, the weak disorder assumption hence allows one to define an intermediate scale $\ell$ where matters simplify.

It is tempting to try to apply Proposition 3.9 repeatedly in order to show that $\boldsymbol{Q} : \Omega \to \mathcal{E}$ is diffusive. To this end, denote the average densities of the environments at scale $l \in \mathbb{N}$ by

$$\bar{q}_l(y) \;:=\; (\mathbb{E}\,\mathrm{RG}_l(\boldsymbol{Q}))_0(0,y)\,,$$

and define an auxiliary function $\delta_\lambda$ acting on integrable functions $\rho : \mathbb{R}^d \to \mathbb{R}$ by

$$\delta_\lambda(\rho) \;:=\; \inf\bigl\{K' > 0 : (70) \text{ apply with } \rho \text{ and } K' \text{ in place of } \bar{q} \text{ and } K, \text{ respectively}\bigr\}\,,$$

with the convention $\delta_\lambda(\rho) = \infty$ provided (70) does not hold for any finite $K'$. Now, let us a call an increasing sequence of scaling parameters $(l_n) \subset \mathbb{N}$ $\boldsymbol{Q}$-*admissible* provided, the ratios $\ell_n := l_n/l_{n-1}$ are integers, and the conditions of Proposition 3.9, are satisfied:

$$\tilde{\ell}_\lambda(\delta_\lambda(\bar{q}_{l_{n-1}})) \;\le\; \ell_n \;\le\; \mathbb{D}[\mathrm{RG}_{l_{n-1}}(\boldsymbol{Q})]^{-1/\mu}\,. \tag{73}$$

Given a $\boldsymbol{Q}$-admissible sequence $(l_n)$, we may express the random environments $\boldsymbol{Q}_n := \mathrm{RG}_{l_n}(\boldsymbol{Q})$ recursively using the semigroup property of the RG transform:

$$\begin{aligned} \boldsymbol{Q}_n \;&=\; \mathrm{RG}_{\ell_n}(\boldsymbol{Q}_{n-1})\,, \qquad n \in \mathbb{N} \\ \boldsymbol{Q}_0 \;&=\; \boldsymbol{Q}\,. \end{aligned} \tag{74}$$

Assuming (73) Proposition 3.9 could be now applied repeatedly to show that the disorder disappears in the limit $n \to \infty$.

By making additional assumptions on $\boldsymbol{Q}$ it is indeed possible to show that a $\boldsymbol{Q}$-admissible sequences $(l_n)$ exist, and the main result of paper (B) follows.

THEOREM 3.10. *Suppose $\boldsymbol{P} : \Omega \to \mathcal{E}(\mathbb{Z}^d)$ is statistically translation invariant and isotropic (cf. Definition 3.2). Moreover, suppose that the random walk on $\mathbb{E}\,\boldsymbol{P}$ is non-trivial, aperiodic, and has exponentially decaying tails, i.e., $\sum_y \mathrm{e}^{\bar{\gamma}|y|}\mathbb{E}\,P_0(0,y) < \infty$, $\bar{\gamma} > 0$. Let $\boldsymbol{Q} : \Omega \to \mathcal{E}(\mathbb{R}^d)$ be generated from $\boldsymbol{P}$ by* (54).

*If the disorder of $\boldsymbol{Q}$ is sufficiently small,*

$$\mathbb{D}[\boldsymbol{Q}] \equiv \mathbb{D}_{\alpha,\beta,\gamma,\varpi}[\boldsymbol{Q}] \;\le\; \ell_1^{-\mu}\,,$$

*for some $\alpha, \beta > 0$, $0 < \gamma < \bar{\gamma}$, and $\varpi \ge 2$, then there exist $\boldsymbol{Q}$-admissible sequences $(l_n)$, such that the environments $\boldsymbol{Q}_n \equiv \mathrm{RG}_{l_n}(\boldsymbol{Q})$ converge to a non-random Gaussian fixed point, in the sense that*

$$\mathbb{D}[\boldsymbol{Q}_n] \;\le\; \ell_n^{-\nu n} \cdot \mathbb{D}[\boldsymbol{Q}]\,, \quad n \in \mathbb{N}\,, \qquad \textit{and} \qquad \bar{q}_{l_n} \xrightarrow{w} \mathrm{N}_{\sigma_*^2 I}\,,$$

*where $\mu, \nu > 0$ are from Proposition 3.9. Moreover, the componentwise variance $\sigma^2 := \sum_y |y^1|^2 \bar{\omega}_y$ is weakly perturbed,*

$$|\sigma_* - \sigma| \;\lesssim\; \mathbb{D}[\boldsymbol{Q}]\,, \tag{75}$$

*and the limit $\sigma^*$ is independent of the choice of the sequence $(l_n)_{n\in\mathbb{N}}$. Moreover, there exists $\ell \ge 3$ such that $l_n := \ell^n$ defines an admissible sequence.*

The theorem is proven by induction on $n$ using the recursion relation (74). The induction hypothesis consists of the statement $\mathbb{D}[\boldsymbol{Q}_n] \leq \varepsilon_n := l_n^{-\nu}\mathbb{D}[\boldsymbol{Q}]$, together with a carefully chosen set of estimates on $\bar{q}_{l_n}$ which guarantee that the estimates (73) apply at each scale $n$, and consequently Proposition 3.9 remains applicable. The guiding principle for finding the right estimates for $\bar{q}_{l_n}$'s is that we expect them to resemble normal densities. In fact, in paper (B) we show that $l_n := \ell^n$ is admissible, so that

$$\sup_{l\in\mathbb{N}_0} \delta_\lambda(\bar{q}_{\ell^n}) \;<\; \infty\,, \tag{76}$$

because $\bar{q}_{\ell^n}$'s converge to a normal density. Actually, at each induction step one obtains

$$\delta_\lambda(\bar{q}_{l_{n+1}}) \;=\; \delta_\lambda(\bar{q}_{l_n}) \,+\, \mathcal{O}(l_n^{-\nu}\varepsilon)\,, \qquad \text{and} \qquad \sigma_{n+1} \;=\; \sigma_n \,+\, \mathcal{O}(l_n^{-\nu}\varepsilon)\,,$$

where $\sigma_n$ is the componentwise variance of $\bar{q}_{l_n}$. In RG-language these recursive corrections to $\sigma_n$'s can be viewed as perturbations caused by the disorder on the trajectory of the RG-flow towards its fixed point. By telescoping one obtains then (75).

Since the uniqueness of the limit in Theorem 3.10 is not proven in paper (B) we provide the proof of this intuitively obvious claim here.

Proof of uniqueness of $\sigma_*$ (sketch). Suppose $(l_n)$ and $(l'_n)$ are two sequences satisfying the necessary conditions with the limits $\sigma_*$ and $\sigma'_*$, respectively. Let us first write

$$|\sigma'_* - \sigma_*| \;\leq\; |\sigma'_* - \sigma'_n| \,+\, |\sigma'_n - \sigma_n| \,+\, |\sigma_n - \sigma_*|\,.$$

By taking $n$ sufficiently large, the first and the last term on the right can be made arbitrarily small. Hence, we must only show that $\lim_n|\sigma'_n - \sigma_n| = 0$.

Denote $\boldsymbol{Q}_n := \mathrm{RG}_{l_n}(\boldsymbol{Q})$, $\boldsymbol{Q}'_n := \mathrm{RG}_{l'_n}(\boldsymbol{Q})$ and $\varepsilon := \mathbb{D}[\boldsymbol{Q}]$. Then Theorem 3.10 yields

$$\mathbb{D}[\boldsymbol{Q}_n] \;\leq\; l_n^{-\nu}\varepsilon\,, \qquad \text{and} \qquad \mathbb{D}[\boldsymbol{Q}'_n] \;\leq\; (l'_n)^{-\nu}\varepsilon\,. \tag{77}$$

The proof is now completed by expressing $\tilde{\boldsymbol{Q}} := \mathrm{RG}_{l_n l'_n}(\boldsymbol{Q})$ and the associated variance $\tilde{\sigma}$ of $\tilde{\boldsymbol{Q}} := \mathrm{RG}_{l_n l'_n}(\boldsymbol{Q})$ in two alternative ways. Namely, we have $\tilde{\boldsymbol{Q}} = \mathrm{RG}_{l'_n}(\boldsymbol{Q}_n)$ and $\tilde{\boldsymbol{Q}} = \mathrm{RG}_{l_n}(\boldsymbol{Q})'_n$. By applying the theorem again on these two representations yields

$$\sigma_n \;=\; \tilde{\sigma} \,+\, \mathcal{O}(\mathbb{D}[\boldsymbol{Q}'_n]) \qquad \text{and} \qquad \sigma'_n \;=\; \tilde{\sigma} \,+\, \mathcal{O}(\mathbb{D}[\boldsymbol{Q}_n])\,,$$

respectively. Combining these bounds with (77) gives the final estimate

$$|\sigma'_n - \sigma_n| \;\leq\; \mathcal{O}(l_n^{-\nu}\varepsilon) + \mathcal{O}((l'_n)^{-\nu}\varepsilon)\,,$$

which converges to zero as $n$ grows. □

Using Theorem 3.10 it is not difficult to show that the induced path measures $\mathsf{P}^{\boldsymbol{Q}_n}$, with the expectations $\mathsf{E}^{\boldsymbol{Q}_n}$, satisfy a quenched invariance principle in the sense that for any continuous functional $f$ acting on continuous functions $u : [0,1] \to \mathbb{R}^d$:

$$\mathsf{E}^{\boldsymbol{Q}_n(\omega)} f(L_n^{-1/2} X_{\lfloor L_n \bullet \rfloor}) \;\to\; E\, f(\sigma_* W)\,, \qquad \text{for } \mathbb{P}-\text{a.e. } \omega \in \Omega\,. \tag{78}$$

Here $(L_n) \subset \mathbb{N}$ is an increasing sequence approaching infinity, and $W : [0,1] \to \mathbb{R}^d$ is a standard Brownian motion under expectation $E$.

It is worth noting that $\mathbb{D}[\boldsymbol{Q}]$ can be written in terms of the environment $\boldsymbol{P}$ on $\mathbb{Z}^d$. Besides, some insignificant constant factors one merely needs to replace the integrals

and $Q_k(x,y)$'s by corresponding sums and $P_k(x,y)$'s in Definition 3.8, respectively. An invariance principle for random walks on the original environment $\boldsymbol{P} : \Omega \to \mathcal{E}(\mathbb{Z}^d)$ is exposed by the following almost sure limit:

$$\mathsf{E}^{\boldsymbol{P}(\omega)} f\big((\ell^n L_n)^{-1/2} X_{\ell^n \lfloor L_n \bullet \rfloor}\big) / \, \mathsf{E}^{\boldsymbol{Q}_n(\omega)} f(L_n^{-1/2} X_{\lfloor L_n \bullet \rfloor}) \;\to\; 1\,, \qquad \text{as } n \to \infty\,.$$

Note that the RG-scheme permits us to directly consider the convergence only along the subset of times

$$\cup_{(l_n)} \cup_{n \in \mathbb{N}_0} \{0, l_n, 2l_n, \ldots, L_n(l_n)\, l_n\} \subset \mathbb{N}_0\,,$$

where $(l_n)$ are $\boldsymbol{Q}$-admissible. Although, one certainly expects that (78) continues to hold if the right hand side is replaced by $\mathsf{E}^{\boldsymbol{P}(\omega)} f(l^{-1/2} X_{\lfloor l \bullet \rfloor})$, there are some difficulties related to the weak decay correlations in space which do not currently permit us to extend the convergence to the full sequence of times (cf. paper (B)).

The assumption that $\boldsymbol{Q}$ corresponds to an environment $\boldsymbol{P}$ on $\mathbb{Z}^d$ is not fundamental. It seems that the theorem could be proven for any continuous environment which has densities, and these densities are almost surely uniformly bounded so that $Q_k(x,y) \leq C_0$ for some non-random constant $C_0$. In the context of static environments, comparison between Theorems 3.10 and 3.5 reveals that by introducing a constant drift in one direction (time), where correlations decay in an integrable way, very weak assumptions concerning the correlations in other directions suffice to prove quenched CLTs.

## 3.4 Generalizations and Fourier's law

The most obvious direction to generalize Theorem 3.10 would be to lift the assumptions concerning the integrability of the correlations of $Q_k$'s in time $k$. Obviously, one expects that the lesser temporal decay should be compensated by stronger assumptions on the spatial decay. The natural question is to then find out the general relationship between the temporal and the spatial decay properties which would guarantee $\boldsymbol{Q} \in \mathcal{G}$ in the perturbative regime. It seems that the approach of paper (B) can be used at least partially to provide results in these directions. It even appears possible to treat true frozen environments by modifying the measure of disorder $\mathbb{D}[\,\bullet\,]$, appropriately. Especially, it would be a very attractive idea to generalize [5] for environments where the spatial decay of correlations in the environment is less than exponential.

RG ideas are not restricted to RWRE setting. After all they have been originally introduced (cf. [17]) in statistical and particle physics context in order to explain the existence of universal scaling limits for a great variety of natural phenomena, as well as to make sense of seemingly divergent mathematical expressions. In particular, in the case of coupled mechanical oscillators it could be possible to define an RG-transform acting on the associated Hamiltonian flow and its random initial condition. By analyzing this transform, one could then attempt to deduce the convergence to a fixed point, which has the property that the evolution of the generated energy density satisfies autonomously a non-linear heat equation.

In the two recent papers [7] and [25] RG-techniques originally introduced in [5] and [6] to deal with RWRE-problems, have been extended with the aim of proving diffusion on realistic classical and quantum Hamiltonian systems. Below, we will briefly comment on how these results relate to the ideas of this section, and well as to the problem of proving Fourier's law from the first principles.

Let $\boldsymbol{Q} : \Omega \to \mathcal{E}(\mathbb{Z}^d)$, suppose that $\mathbb{E}\,\boldsymbol{Q}(0,y) = \frac{1}{2d}$, when $|x-y| = 1$ and $\mathbb{E}\,\boldsymbol{Q}(0,y) = 0$ otherwise. Let us define a field $P = (P_k)_{k\in\mathbb{N}_0}$ of occupation probabilities $P_k : \mathbb{Z}^d \to [0,1]$ by $P_k := \mathsf{P}^{\boldsymbol{Q}}\{X_k \in \bullet\}$. From these definitions it follows that the field $P$ satisfies a linear discrete heat equation with a random perturbation

$$P_{k+1} - P_k \;=\; \tfrac{1}{2d}\,\Delta P_k \;+\; P_k B_k\,, \tag{79}$$

where $Q_k = \mathbb{E}\,Q_k + B_k$, and the terms on the right are specified by

$$(\Delta P)(x) = \sum_{y\in\mathbb{Z}^d:|y-x|=1} (P(y) - P(x))\,, \qquad PB_k(y) := \sum_{x\in\mathbb{Z}^d} P(x)\,B_k(x,y)\,.$$

The regular CLT now corresponds to the limit where $l^{d/2}P_{lk}(l\,\bullet)$ converges to a normal density with variance of size $\mathcal{O}(k)$.

Now, in paper [7] an integrable field $E = (E_k)_{k\in\mathbb{N}_0}$, $E_k : \mathbb{Z}^d \to [0,\infty)$, with $\|E_0\|_1 := \sum_x E_0(x) < \infty$, solving a non-linear version of (79) is studied. Namely, the linear random perturbation $E_kB_k$ is replaced by a *non-linear* locally conservative random perturbation $\beta_k(E_k)$, so that $\sum_x \beta_k^\omega(F;x) = 0$ for $\mathbb{P}$-a.e. $\omega$, and $\beta_k(F) \to 0$ as $|x| \to \infty$, for any integrable field $F$ on $\mathbb{Z}^d$.

The presence of the non-linearity in the random heat equation implies that $E_k(x)$'s can not be interpreted as probabilities even if one normalizes $\|E_0\|_1 = 1$, and thus the model can not be considered as RWRE. Nevertheless, provided the perturbation $\beta_k$ is sufficiently weak it can be shown by applying the RG methods adapted from [5] and [6] that the scaled field

$$E_k(x) \mapsto l^{d/2}E_{lk}(lx) := E_k^{(l)}(x)\,, \tag{80}$$

satisfies a linear heat equation without perturbations in the scaling limit $l \to \infty$. In the paper [7] the field $E$ arises as a simplified version of the local energy density generated by a deterministic but non-Hamiltonian coupled map lattice type dynamical system. Hence, the dynamical 'random environment' for $E$ is actually a chaotic dynamical system started from a random initial state.

The physical interpretation of the result of paper [7] is the following: An infinitely large system of oscillators is started from a random initial state where a finite amount of energy is concentrated near origin. As time passes, this energy spreads diffusively to the whole lattice. At very large times the energy density is essentially zero everywhere and hence the conductivity $\kappa(E)$ in the heat equation can be replaced by a constant value $\kappa(0)$, and a linear heat equation results.

In another recent paper [25] an RG scheme very similar to the one used in paper (B) is used to deal with a purely quantum mechanical, i.e., purely unitary, system consisting of a massive particle interacting with an quantized electromagnetic field which is started from a Gibbs state. The main result of this paper is that the location of the particle diffuses. Thus the results in [25] constitute one of the very few rigorous results in which a purely Hamiltonian evolution has been shown to give rise to a diffusive behavior without making unphysical simplifications. Besides being purely quantum mechanical, this model does not fit into regular RWRE framework either, also because the particle is able to affect its environment.

As for future projects it would be very interesting to further develop the results of the papers [7] and [25], and try to isolate exact conditions when the RG approach can

be applied effectively. Besides, attempting to use the methods on more realistic models of heat conduction, it would also be interesting to apply the related RG-techniques to other RWRE-like problems, e.g., weakly self-avoiding random walks, where the walker is also able affect its environment.

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